STARLIKENESS OF CERTAIN INTEGRAL OPERATORS AND PROPERTIES OF A SUBCLASS OF CLOSE-TO-CONVEX FUNCTIONS

CHUNG YAO LIANG

UNIVERSITI SAINS MALAYSIA

2017

STARLIKENESS OF CERTAIN INTEGRAL OPERATORS AND PROPERTIES OF A SUBCLASS OF CLOSE-TO-CONVEX FUNCTIONS

by

CHUNG YAO LIANG

Thesis submitted in fulfilment of the requirements for the degree of Master of Science

February 2017

ACKNOWLEDGEMENT

I take this opportunity to express my deep sense of gratitude to my supervisor, Assoc. Prof. Dr. Lee See Keong, for his constant encouragement and invaluable guidance in the successful completion of this dissertation. Indeed, his critical observations and constructive criticisms have enabled me to present the work in this form. I wish to record my thanks to my co-supervisor Dr. Maisarah Haji Mohd for her valuable advice and support during my studies. I also express my gratitude to Prof. Hailiza Kamarulhaili, Dean of School of Mathematical Sciences, Universiti Sains Malaysia, for providing necessary facilities during the course of my study.

I would like to place on record my deep sense of gratitude to the Ministry of Higher Education of Malaysia as well as Universiti Sains Malaysia for providing me the much needed financial support in the form of a student Fellowship as well as access to the latest research that enable me to pursue my higher studies.

Last but not least, I am sincerely thankful to my parents who have always been a guiding star and a constant source of inspiration and encouragement for me. I also acknowledge the moral and financial support of my family members who have always backed me up and motivated me to achieve my goals.

TABLE OF CONTENTS

LIST OF SYMBOLS

$A[f]$	Alexander operator
$\mathcal{A}_n$	Class of all normalized analytic functions f of the form $f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k, \quad z \in \mathbb{D}$
$\mathcal{A}$	$\mathcal{A}_1$
$\mathbb{C}$	Complex plane
$\mathcal{C}$	Class of convex functions in $\mathcal{A}$
$\mathcal{C}(\alpha)$	Class of convex functions of order α in $\mathcal{A}$
$\mathbb{D}$	Unit disk
$\mathcal{H}$	Class of all analytic functions in $\mathbb{D}$
$\mathcal{H}[a,n]$	Class of all analytic functions f of the form $f(z) = a + \sum_{k=n}^{\infty} a_k z^k, \quad z \in \mathbb{D}$
Im	Imaginary part of a complex number
k	Koebe function, $k(z) = z/(1-z)^2$
$\mathcal{K}$	Class of close-to-convex functions in $\mathcal{A}$
$L[f]$	Libera operator
$L_\gamma[f]$	Bernardi operator
m	Möbius function, $m(z) = (1+z)/(1-z)$
max	Maximum
$\mathbb{N}$	Natural numbers
$\mathcal{P}$	Class of normalized analytic function with positive real part
$\Re$	Real part of a complex number
$\mathcal{S}$	Class of all normalized univalent functions f in $\mathcal{A}$
$\mathcal{S}^*$	Class of starlike functions in $\mathcal{A}$
$\mathcal{S}^*(\alpha)$	Class of starlike functions of order α in $\mathcal{A}$
$\mathcal{S}_s^*$	Class of starlike functions with respect to symmetric points in $\mathcal{A}$
ω	Schwarz function
$\prec$	Subordinate to

KEBAKBINTANGAN BEBERAPA PENGOPERASI KAMIRAN DAN SIFAT SUBKELAS FUNGSI HAMPIR CEMBUNG

ABSTRAK

Disertasi ini mengkaji syarat cukup bagi fungsi analisis bernilai kompleks bak-bintang dalam cakera unit dan ciri-ciri suatu subkelas fungsi hampir cembung. Suatu kajian ringkas mengenai konsep asas dan keputusan dari teori fungsi univalent analitik telah diberikan. Syarat cukup bagi fungsi analitik yang tertakrif dalam cakera unit untuk menjadi bak-bintang peringkat β yang mematuhi ketidaksamaan pembezaan ketiga. Dengan menggunakan ketidaksamaan pembezaan ketiga, kebakbintangan suatu pengoperasi kamiran akan diperoleh. Keputusan yang diperoleh menyatukan hasil kajian terdahulu. Tambahan pula, suatu subklass fungsi hampir cembung yang baru telah diperkenalkan dan beberapa keputusan menarik telah diperoleh seperti sifat rangkuman, anggaran ketidaksamaan Fekete-Szego bagi fungsi tergolong dalam klass, anggaran pekali, dan syarat cukup.

STARLIKENESS OF CERTAIN INTEGRAL OPERATORS AND PROPERTIES OF A SUBCLASS OF CLOSE-TO-CONVEX FUNCTIONS

ABSTRACT

The present dissertation investigates the sufficient conditions for an analytic function to be starlike in the open unit disk $\mathbb{D}$ and some properties of certain subclass of close-to-convex functions. A brief survey of the basic concepts and results from the classical theory of analytic univalent functions are given. Sufficient conditions for analytic functions satisfying certain third-order differential inequalities to be starlike in $\mathbb{D}$ is derived. As a consequence, conditions for starlikeness of functions defined by triple integral operators are obtained. Connections are also made to earlier known results. Furthermore, a new subclass of close-to-convex functions is introduced and studied. Some interesting results are obtained such as inclusion relationships, an estimate for the Fekete-Szegö functional for functions belonging to the class, coefficient estimates, and a sufficient condition.

CHAPTER 1

INTRODUCTION

1.1 A Short History

Geometric function theory is a branch of complex analysis, which studies the geometric properties of analytic functions. The theory of univalent functions is one of the most important subjects in geometric function theory. The study of univalent functions was initiated by Koebe [21] in 1907. One of the major problems in this field had been the Bieberbach [4] conjecture dating from the year 1916, which asserts that the modulus of the nth Taylor coefficient of each normalized analytic univalent function is bounded by n. The conjecture was not completely solved until 1984 by French-American mathematician Louis de Branges [9].

1.2 Basic Definitions And Properties Of The Class Of Univalent Functions

Let $\mathbb{C}$ be the complex plane of complex numbers. A *domain* is an open connected subset of $\mathbb{C}$. A domain is said to be *simply connected* if its complement is connected. Geometrically, a simply connected domain is a domain without any holes in it. A complex-valued function f of a complex variable is said to be *differentiable* at a point $z_0 \in \mathbb{C}$ if it has a derivative

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

at z_0. The function f is *analytic* at z_0 if it is differentiable at every point in some neighborhood of z_0. It is one "miracle" of complex analysis that an analytic function f must have derivatives of all order at z_0 and has a Taylor series expansion

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \quad a_n = \frac{f^{(n)}(z_0)}{n!},$$

which converges in some open disk centered at z_0. It is analytic in a domain if it is analytic at every point of the domain.

Definition 1.1. [15] A function f on $\mathbb{C}$ is said to be *univalent* (*one-to-one*) in a domain $\mathcal{D} \subset \mathbb{C}$ if for $z_1, z_2 \in \mathcal{D}$,

$$f(z_1) = f(z_2) \Rightarrow z_1 = z_2,$$

or equivalently

$$z_1 \neq z_2 \Rightarrow f(z_1) \neq f(z_2).$$

A function f is said to be *locally univalent* at a point $z_0 \in \mathcal{D}$ if it is univalent in some neighborhood of z_0. For analytic functions f, the condition $f'(z_0) \neq 0$ is equivalent to local univalence at z_0. A function f univalent in a domain $\mathcal{D}$ is locally univalent at each of the points in $\mathcal{D}$, but the converse is not true in general. For example, consider the function $f(z) = z^2$ in the domain $\mathbb{C} - \{0\}$. Since $f'(z) = 2z \neq 0$ for $z \neq 0$, it follows that $f(z) = z^2$ is locally univalent in $\mathbb{C} - \{0\}$. But $f(-z) = (-z)^2 = z^2 = f(z)$, so this function is not univalent in the whole domain $\mathbb{C} - \{0\}$. However, $f(z) = z^2$ is univalent on $\{z \in \mathbb{C} : \Re z > 0\}$. (Here, $\Re z$ denote the real part of z.)

Noshiro [36] and Warschawski [56] independently provides a sufficient condition for an analytic function to be univalent in a convex domain $\mathcal{D}$, which is now known as the Noshiro-Warschawski Theorem. A domain $\mathcal{D}$ is *convex* if the line segment joining any two points in $\mathcal{D}$ lies completely in $\mathcal{D}$, that is, for every $z_1, z_2 \in \mathcal{D}$, we have $z_1 + t(z_2 - z_1) \in \mathcal{D}$ for $0 \leq t \leq 1$. Examples of convex domain are circular disk and half-plane.

Theorem 1.1. (Noshiro-Warschawski Theorem) [36, 56] If f is analytic in a convex domain $\mathcal{D}$, and $\Re\{f'\} > 0$ in $\mathcal{D}$, then f is univalent in $\mathcal{D}$. (Here, $\Re\{f'\}$ denote the real part of f'.)

Proof. We will show that $f(z_1) \neq f(z_2)$ for all $z_1, z_2 \in \mathcal{D}$ with $z_1 \neq z_2$. Choose distinct points $z_1, z_2 \in \mathcal{D}$. Since $\mathcal{D}$ is a convex domain, the straight line segment $z = z_1 + t(z_2 - z_1), 0 \leq t \leq 1$, must lie in $\mathcal{D}$. By integrating along this line segment from z_1 to z_2, we have

$$f(z_2) - f(z_1) = \int_{z_1}^{z_2} f'(z)\, dz = \int_0^1 f'\Big(z_1 + t(z_2 - z_1)\Big)(z_2 - z_1)\, dt.$$

Dividing by $z_2 - z_1$ and taking the real part, we get

$$\Re\left\{\frac{f(z_2) - f(z_1)}{z_2 - z_1}\right\} = \Re\left\{\int_0^1 f'\Big(z_1 + t(z_2 - z_1)\Big)\, dt\right\}.$$

Since f is analytic in $\mathcal{D}$, f' exists and is analytic in $\mathcal{D}$. It is known that an analytic function is differentiable and continuous in $\mathcal{D}$. It follows that

$$\Re\left\{\int_0^1 f'\Big(z_1 + t(z_2 - z_1)\Big)\, dt\right\} = \int_0^1 \Re\left\{f'\Big(z_1 + t(z_2 - z_1)\Big)\right\} dt.$$

Since $\Re\{f'\} > 0$ for all $z \in \mathbb{D}$, it follows that

$$\Re\left\{\frac{f(z_2)-f(z_1)}{z_2-z_1}\right\} = \int_0^1 \Re\left\{f'\Big(z_1+t(z_2-z_1)\Big)\right\} dt > 0.$$

Hence,

$$\frac{f(z_2)-f(z_1)}{z_2-z_1} \neq 0$$

and so $f(z_1) \neq f(z_2)$. $\square$

Let $\mathcal{H}$ denote the class of all analytic functions in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. For a positive integer n and $a \in \mathbb{C}$, let

$$\mathcal{H}[a,n] = \left\{ f \in \mathcal{H} : f(z) = a + \sum_{k=n}^{\infty} a_k z^k,\ z \in \mathbb{D} \right\}$$

and

$$\mathcal{A}_n = \left\{ f \in \mathcal{H} : f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k,\ z \in \mathbb{D} \right\},$$

with $\mathcal{A}_1 := \mathcal{A}$. So, $\mathcal{A}$ is the class of analytic functions in $\mathbb{D}$ with normalization $f(0) = 0$ and $f'(0) = 1$. The subclass of $\mathcal{A}$ consisting of univalent functions is denoted by $\mathcal{S}$.

Example 1.1. An important example of functions in the class $\mathcal{S}$ is the Koebe function, given by

$$k(z) = \frac{z}{(1-z)^2} = \sum_{n=1}^{\infty} n z^n = z + 2z^2 + 3z^3 + \cdots .$$

It is easy to verify that the Koebe function is analytic, normalized and univalent in $\mathbb{D}$. Since the Koebe function is differentiable at every $z \in \mathbb{D}$, it follows that Koebe function is analytic in $\mathbb{D}$. Also, the Koebe function satisfies the condition $k(0) = 0$ and

$k'(0) = 1$ where $k'(z) = (1+z)/(1-z)^3$. Hence, the Koebe function is normalized in $\mathbb{D}$. To see that the Koebe function is univalent in $\mathbb{D}$, suppose that $k(z_1) = k(z_2)$, that is,

$$\frac{z_1}{(1-z_1)^2} = \frac{z_2}{(1-z_2)^2}, \quad z_1, z_2 \in \mathbb{D}.$$

After a simple computation, we get

$$(z_1 - z_2)(1 - z_1 z_2) = 0.$$

Since $z_1, z_2 \in \mathbb{D}$, we have $|z_1| < 1$ and $|z_2| < 1$ and therefore $|z_1 z_2| = |z_1||z_2| < 1$. This shows that $1 - z_1 z_2 \neq 0$ in $\mathbb{D}$. Thus we must have $z_1 - z_2 = 0$, that is, $z_1 = z_2$. So, the Koebe function, k is univalent in $\mathbb{D}$.

Geometrically, the Koebe function maps $\mathbb{D}$ univalently onto the entire complex plane minus the negative axis from $-1/4$ to infinity. This can be seen by observing that the Koebe function can be written as a composition of three univalent analytic functions, that is,

$$(u_3 \circ u_2 \circ u_1)(z) = \frac{1}{4}\left[\left(\frac{1+z}{1-z}\right)^2 - 1\right] = \frac{z}{(1-z)^2},$$

where

$$u_1(z) = \frac{1+z}{1-z}, \quad u_2(z) = z^2, \quad \text{and} \quad u_3(z) = \frac{1}{4}[z-1].$$

It is easy to see that u_1, u_2 and u_3 are analytic and they map univalently on this composition. Since u_1 is the quotient of two analytic functions $1+z$ and $1-z$, therefore it is analytic in $\mathbb{D}$. To see that u_1 is univalent in $\mathbb{D}$, suppose that $u_1(z_1) = u_1(z_2)$, that is,

$$\frac{1+z_1}{1-z_1} = \frac{1+z_2}{1-z_2}, \quad z_1, z_2 \in \mathbb{D}.$$

After simplifying, we obtain $z_1 - z_2 = 0$ or $z_1 = z_2$. Hence, the function $u_1(z) = (1+z)/(1-z)$ is univalent in $\mathbb{D}$. We have

$$\Re\{u_1(z)\} = \Re\left\{\frac{1+z}{1-z}\right\} = \frac{1}{2}\left(\frac{1+z}{1-z} + \overline{\frac{1+z}{1-z}}\right) = \frac{1}{2}\left(\frac{1+z}{1-z} + \frac{1+\bar{z}}{1-\bar{z}}\right) = \frac{1-|z|^2}{|1-z|^2} > 0$$

for $|z| < 1$. Since $u_1(0) = 1$, it follows that $\mathbb{D}$ is mapped univalently onto the right half-plane, $\{z \in \mathbb{C} : \Re\{z\} > 0\}$, under the mapping $u_1(z) = (1+z)/(1-z)$.

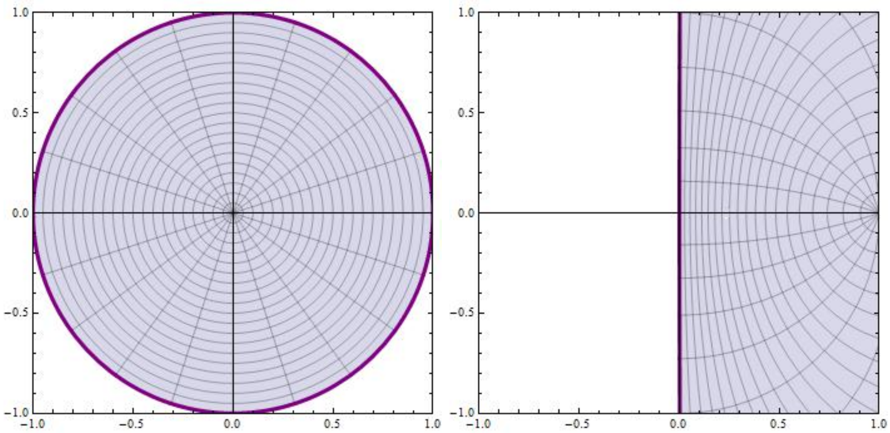

Figure 1.1: The image of unit disk $\mathbb{D}$ under the mapping $u_1(z) = (1+z)/(1-z)$. ■

Since u_2 is the product of two analytic functions z, it follows that u_2 is analytic in the right half plane (a convex domain). For $u_2(z) = z^2$, $\Re\{z\} > 0$, we have

$$\Re\{u_2'(z)\} = 2\Re\{z\} > 0.$$

Hence, by Noshiro - Warschawski Theorem (Theorem 1.1), the function $u_2(z)$ is univalent in the right half plane. Note that the upper right half plane is mapped onto upper half plane, positive real axis is mapped onto positive real axis and the lower right half plane is mapped onto lower half plane. Note that $u_2(0) = 0$ and the imaginary axis is mapped onto the negative real axis. Since the origin and the imaginary axis lies outside of the right half plane, it follows that the function u_2 mapped the right half plane univalently onto the entire complex plane minus the nonnegative real axis.

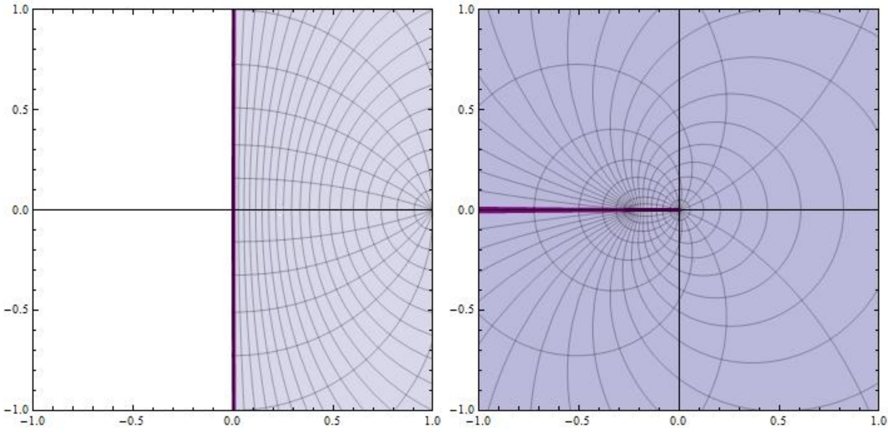

Figure 1.2: The image of right half plane under the mapping $u_2(z) = z^2$.

Clearly, u_3 is analytic in entire complex plane minus the nonnegative real axis. To see that u_3 is univalent, suppose that $u_3(z_1) = u_3(z_2)$, that is,

$$\frac{1}{4}(z_1 - 1) = \frac{1}{4}(z_2 - 1).$$

After simplifying, we obtain $z_1 - z_2 = 0$ or $z_1 = z_2$. Hence, u_3 is univalent in entire complex plane minus the nonnegative real axis. So, u_3 translates the nonnegative real axis one space to the left and multiplies by a factor of $1/4$. Therefore, u_3 maps the entire complex plane except for the nonnegative real axis univalently onto the entire complex plane minus the negative axis from $-1/4$ to infinity.

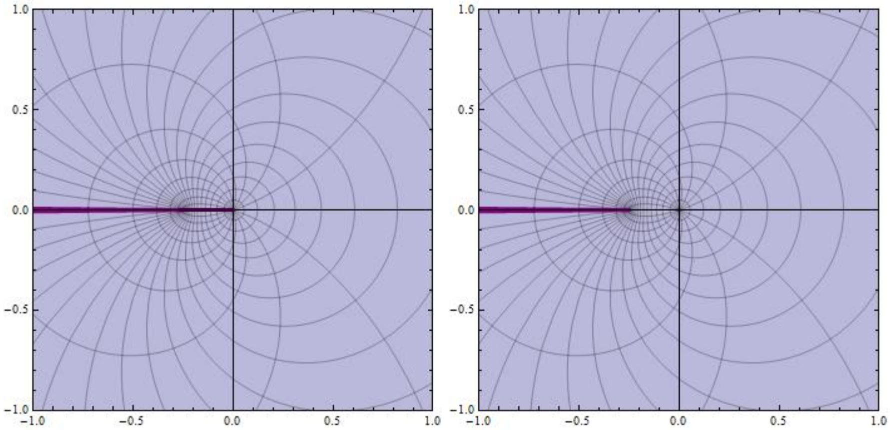

Figure 1.3: The image domain under the mapping $u_3(z) = \frac{1}{4}(z-1)$.

For every function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ in $\mathcal{S}$, Bieberbach [4] showed that the second coefficient a_2 of the series expansion is bounded by 2, which is now known as Bieberbach's Theorem.

Theorem 1.2. [4] (Bieberbach's Theorem) If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}$, then $|a_2| \leq 2$, with equality if and only if f is a rotation of the Koebe function.

The extremal property of the Koebe function tempted Bieberbach [4] to conjecture that $|a_n| \leq n$ holds for all f in $\mathcal{S}$. This conjecture was popularly known as Bieberbach's conjecture.

Conjecture 1.1. [4] (Bieberbach's Conjecture) The coefficients of each function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}$ satisfy $|a_n| \leq n$ for $n = 2, 3, \ldots$. Strict inequality holds for all n unless f is the Koebe function or one of its rotations.

The conjecture had been proven for the case $n = 2, 3, 4, 5, 6$ by some researchers before Louis de Branges [9] proved the general case $|a_n| \leq n$ in 1984. This is summarized in the table below.

Researchers	Result
Bieberbach [4] (1916)	$\lvert a_2 \rvert \leq 2$
Löwner [29] (1923)	$\lvert a_3 \rvert \leq 3$
Garabedian and Schiffer [14] (1955)	$\lvert a_4 \rvert \leq 4$
Pederson [42] (1968), Ozawa [39] (1969)	$\lvert a_6 \rvert \leq 6$
Pederson and Schiffer [41] (1972)	$\lvert a_5 \rvert \leq 5$
de Branges [9] (1984)	$\lvert a_n \rvert \leq n$

Nowadays, the Bieberbach conjecture is also called the de Branges Theorem.

1.2.1 Function With Positive Real Part And Subordination

Definition 1.2. [15] An analytic function of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$$

in $\mathbb{D}$ with $\Re\{p(z)\} > 0$ is called a *function of positive real part* or *Carathéodory function*. The set of all functions of positive real part in $\mathbb{D}$ is denoted by $\mathcal{P}$.

Example 1.2. The Möbius function

$$m(z) = \frac{1+z}{1-z} = 1 + 2z + 2z^2 + \cdots = 1 + 2\sum_{n=1}^{\infty} z^n,$$

is in the class $\mathcal{P}$ since $\Re\{(1+z)/(1-z)\} > 0$, as shown in Example 1.1. ■

Example 1.3. The function

$$w(z) = \frac{1+z^n}{1-z^n}, \quad n = 1, 2, 3, \ldots$$

belongs to $\mathcal{P}$ for $|z| < 1$. To see this, note that $w(0) = 1$. Further, $w(z) = (m \circ \phi)(z)$ where m is the Möbius function and $\phi(z) = z^n$. Since $|\phi(z)| < 1$, it follows from Example 1.2 that $\Re\{w\} > 0$. ■

In 1911, Herglotz [18] obtained an integral formula for functions in the class $\mathcal{P}$.

Theorem 1.3. [18] Let p be an analytic function in $\mathbb{D}$ satisfying $p(0) = 1$. Then $p \in \mathcal{P}$ if and only if

$$p(z) = \int_0^{2\pi} \frac{1 + ze^{-it}}{1 - ze^{-it}}\, d\mu(t),$$

where $d\mu(t) \geq 0$ and $\int_0^{2\pi} d\mu(t) = \mu(2\pi) - \mu(0) = 1$.

The Herglotz formula gives the bounds for the coefficients of functions in $\mathcal{P}$. This result is due to Carathéodory.

Theorem 1.4. [5] If $p \in \mathcal{P}$ with $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$, $z \in \mathbb{D}$, then $|p_n| \leq 2$ for all $n \in \mathbb{N}$. These estimates are sharp.

Proof. Since $p \in \mathcal{P}$, by Theorem 1.3, we have

$$p(z) = \int_0^{2\pi} \frac{1 + ze^{-it}}{1 - ze^{-it}}\, d\mu(t),$$

where $d\mu(t) \geq 0$ and $\int_0^{2\pi} d\mu(t) = \mu(2\pi) - \mu(0) = 1$. Therefore,

$$\begin{aligned} p(z) &= \int_0^{2\pi} \frac{1 + ze^{-it}}{1 - ze^{-it}}\, d\mu(t) \\ &= \int_0^{2\pi} (1 + 2ze^{-it} + 2z^2 e^{-2it} + 2z^3 e^{-3it} + \cdots) d\mu(t) \\ &= 1 + \sum_{n=1}^{\infty} \left(2 \int_0^{2\pi} e^{-int} d\mu(t) \right) z^n. \end{aligned}$$

Now comparing this with $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$ yields

$$p_n = 2\int_0^{2\pi} e^{-int} d\mu(t).$$

Hence,

$$\begin{aligned}|p_n| &= 2\left|\int_0^{2\pi} e^{-int} d\mu(t)\right| \\ &\leq 2\int_0^{2\pi} |e^{-int}||d\mu(t)| \\ &= 2\int_0^{2\pi} d\mu(t) \\ &= 2.\end{aligned}$$

The Möbius function in Example 1.2 showed that the bound $|p_n| \leq 2$ is sharp. $\square$

Closely related to the class $\mathcal{P}$ is the class of functions with positive real part of order $\alpha, 0 \leq \alpha < 1$.

Definition 1.3. [15] An analytic function p with the normalization $p(0) = 1$ in $\mathbb{D}$ is said to be a *function of positive real part of order* α, $0 \leq \alpha < 1$ if $\Re\{p(z)\} > \alpha$. The set of all functions of positive real part of order α is denoted by $\mathcal{P}(\alpha)$. Observe that for $\alpha = 0$, we have $\mathcal{P}(0) = \mathcal{P}$.

Example 1.4. Consider the function $f(z) = 1/(1-z)$, $z \in \mathbb{D}$. Since f is differentiable for all $z \in \mathbb{D}$, it is analytic in $\mathbb{D}$. Clearly, $f(0) = 1$. Furthermore,

$$\Re\left\{\frac{1}{1-z}\right\} = \Re\left\{\frac{1}{2}\left(\frac{1+z}{1-z}+1\right)\right\} = \frac{1}{2}\Re\left\{\frac{1+z}{1-z}\right\} + \frac{1}{2} > 0 + \frac{1}{2} = \frac{1}{2}.$$

Therefore, the function $f(z) = 1/(1-z)$ belongs to $\mathcal{P}(1/2)$.

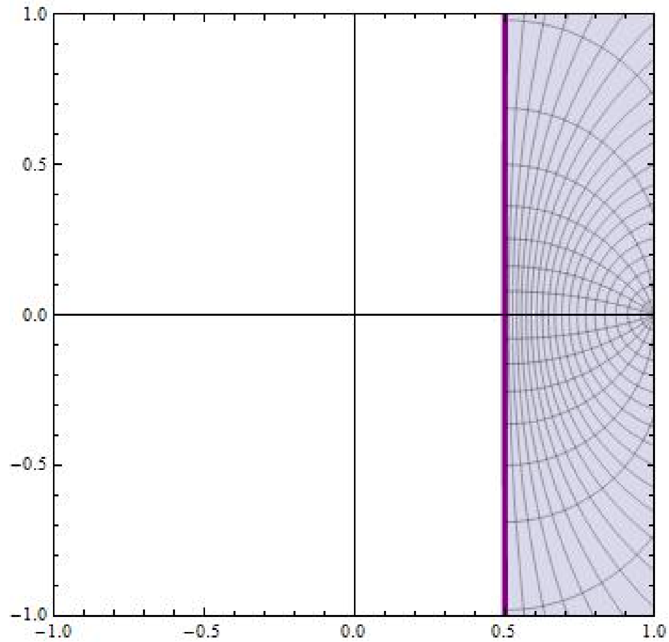

Figure 1.4: The real part of $f(z) = 1/(1-z)$. ■

Example 1.5. The function

$$f(z) = \frac{1+(1-2\alpha)z}{1-z} = (1-\alpha)\left(\frac{1+z}{1-z}\right) + \alpha = 1 + 2(1-\alpha)\sum_{n=1}^{\infty} z^n$$

is in the class $\mathcal{P}(\alpha)$ for $0 \leq \alpha < 1$. Clearly, $f(0) = 1$. Also,

$$\Re\left\{(1-\alpha)\left(\frac{1+z}{1-z}\right) + \alpha\right\} = (1-\alpha)\Re\left\{\frac{1+z}{1-z}\right\} + \alpha > \alpha$$

using the fact that $\Re\left\{(1+z)/(1-z)\right\} > 0$ as in Example 1.1. For $\alpha = 0$, we have the inequality

$$\Re\{f(z)\} = \Re\left\{\frac{1+z}{1-z}\right\} > 0$$

which has been discussed in Example 1.1.

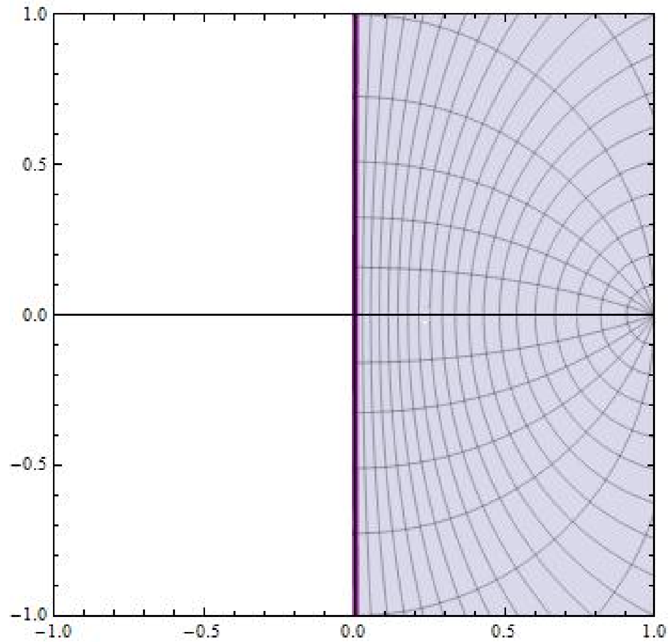

Figure 1.5: The real part of $f(z) = (1+z)/(1-z)$. ■

Definition 1.4. A function ω which is analytic in $\mathbb{D}$ and satisfies the properties $\omega(0) = 0$ and $|\omega(z)| < 1$ is called a *Schwarz function*. The class of all Schwarz functions is denoted by Ω.

Definition 1.5. For analytic functions f and g on $\mathbb{D}$, we say that f is subordinate to g, denoted $f \prec g$, if there exists a Schwarz function ω in $\mathbb{D}$ such that

$$f(z) = g(\omega(z)), \quad z \in \mathbb{D}.$$

Example 1.6. The function z^2 is subordinate to z in $\mathbb{D}$. Referring to Definition 1.5, we can choose $\omega(z) = z^2$. Clearly, ω is analytic in $\mathbb{D}$ and $\omega(0) = 0$. Also, $|\omega(z)| = |z^2| = |z|^2 < 1$ since $z \in \mathbb{D}$. ■

Example 1.7. The function z^4 is subordinate to z^2 in $\mathbb{D}$. Referring to Definition 1.5, we can choose $\omega(z) = z^2$. Clearly, ω is analytic in $\mathbb{D}$ and $\omega(0) = 0$. Also, $|\omega(z)| = |z^2| = |z|^2 < 1$ since $z \in \mathbb{D}$. In general, we have $z^{2n} \prec z^2$ in $\mathbb{D}$ for n a positive integer. ■

Theorem 1.5. Let f and g be analytic in $\mathbb{D}$. If g is univalent in $\mathbb{D}$, then $f \prec g$ if and only if $f(\mathbb{D}) \subset g(\mathbb{D})$ and $f(0) = g(0)$.

Proof. Suppose $f \prec g$. By Definition 1.5, there exists a Schwarz function ω such that $f(z) = g(\omega(z))$. Since $\omega(\mathbb{D}) \subset \mathbb{D}$, it follows that $f(\mathbb{D}) = g(\omega(\mathbb{D})) \subset g(\mathbb{D})$. Also $f(0) = g(\omega(0)) = g(0)$.

Conversely, suppose $f(\mathbb{D}) \subset g(\mathbb{D})$ and $f(0) = g(0)$. Since g is univalent in $\mathbb{D}$, it follows that g maps $\mathbb{D}$ one-to-one onto its image $g(\mathbb{D})$. Therefore, the inverse g^{-1} exists in $g(\mathbb{D})$ and maps $g(\mathbb{D})$ onto $\mathbb{D}$. Since g is analytic in $\mathbb{D}$, the inverse g^{-1} is also analytic in $g(\mathbb{D})$. Since $f(\mathbb{D}) \subset g(\mathbb{D})$, it follows that the function

$$\omega(z) := g^{-1}(f(z))$$

is analytic in $\mathbb{D}$ and $|\omega(z)| < 1$. Thus, we obtain $f(z) = g(\omega(z))$. From this, we have $g(\omega(0)) = f(0) = g(0)$. Since g is univalent, this forces $\omega(0) = 0$ by Definition 1.1. So, ω is a Schwarz function such that $f(z) = g(\omega(z))$ for $z \in \mathbb{D}$. Therefore, $f \prec g$. $\square$

1.2.2 Subclasses Of Univalent Functions

In the course of tackling the Bieberbach conjecture, new classes of analytic and univalent functions were defined and some nice properties of these classes were widely investigated. Examples of such classes are the classes of starlike, convex and close-to-convex functions.

A domain $\mathcal{D} \subset \mathbb{C}$ is said to be *starlike with respect to a point* w_0 in $\mathcal{D}$ if every line joining the point w_0 to every other point w in $\mathcal{D}$ lies entirely inside $\mathcal{D}$. A domain which is starlike with respect to the origin is simply called a *starlike* domain. Geometrically, a starlike domain is a domain whose all points can be seen from the origin. A function $f \in \mathcal{A}$ is called a *starlike* function if $f(\mathbb{D})$ is a starlike domain. The subclass of $\mathcal{S}$ consisting of all starlike functions is denoted by $\mathcal{S}^*$.

Theorem 1.6. [10, Theorem 2.10] Let $f \in \mathcal{A}$. Then $f \in \mathcal{S}^*$ if and only if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > 0, \quad z \in \mathbb{D}.$$

Example 1.8. Recall from Example 1.1, the Koebe function $k(z) = z/(1-z)^2$ is analytic and normalized in $\mathbb{D}$. Moreover, k is in $\mathcal{S}^*$ since

$$\Re\left\{\frac{zk'(z)}{k(z)}\right\} = \Re\left\{\frac{z(1+z)}{(1-z)^3}\frac{(1-z)^2}{z}\right\} = \Re\left\{\frac{1+z}{1-z}\right\} > 0. \qquad \blacksquare$$

Example 1.9. The function

$$f(z) = \frac{z}{1-z^2} = \sum_{n=0}^{\infty} z^{2n+1}$$

is analytic in $\mathbb{D}$ since f is differentiable at all $z \in \mathbb{D}$. Clearly, $f(0) = 0$. Since $f'(z) = (1+z^2)/(1-z^2)^2$, it follows that $f'(0) = 1$. Also,

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} = \Re\left\{\frac{z(1+z^2)}{(1-z^2)^2}\frac{(1-z^2)}{z}\right\} = \Re\left\{\frac{1+z^2}{1-z^2}\right\} > 0.$$

The last inequality follows from Example 1.3. Hence, the function $f(z) = z/(1-z^2)$ is starlike on $\mathbb{D}$.

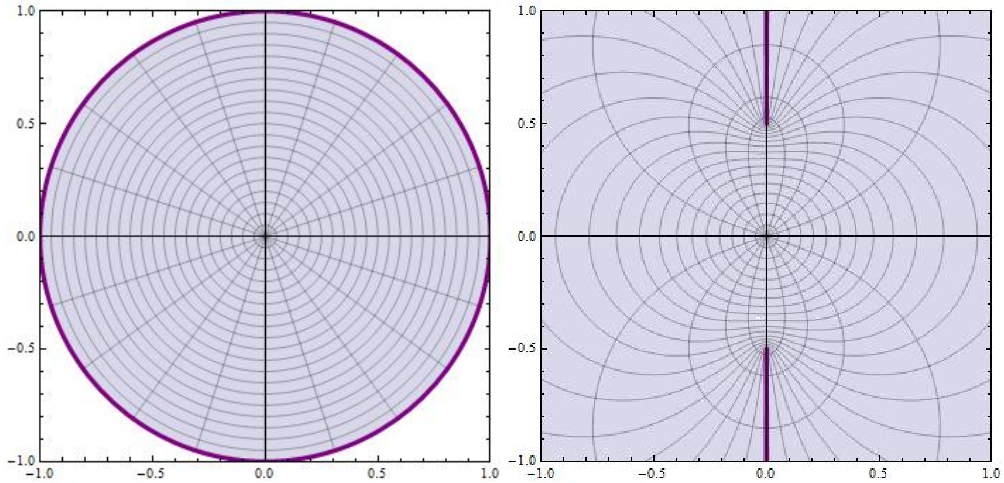

Figure 1.6: The image of $\mathbb{D}$ under the mapping $f(z) = z/(1-z^2)$. ■

A domain $\mathcal{D} \subset \mathbb{C}$ is said to be *convex* if every linear segment joining any two points in $\mathcal{D}$ lies completely inside $\mathcal{D}$. In other words, the domain $\mathcal{D}$ is convex if and only if it is starlike with respect to every point in $\mathcal{D}$. A function $f \in \mathcal{A}$ is said to be *convex* if $f(\mathbb{D})$ is a convex domain. The subclass of $\mathcal{S}$ consisting of all convex functions is denoted by $\mathcal{C}$.

Theorem 1.7. [10, Theorem 2.11] Let $f \in \mathcal{A}$. Then $f \in \mathcal{C}$ if and only if

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0, \quad z \in \mathbb{D}.$$

Example 1.10. The identity function $f(z) = z$ is a convex function. Note that $f''(z) = 1$ and $f''(z) = 0$. Hence,

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} = 1 > 0. \qquad ■$$

Example 1.11. The function

$$f(z) = \frac{z}{1-z} = \sum_{n=1}^{\infty} z^n$$

is analytic in $\mathbb{D}$ since f is differentiable in $\mathbb{D}$. Clearly, $f(0) = 0$. Since $f'(z) = 1/(1-z)^2$, it follows that $f'(0) = 1$. Also, $f''(z) = 2/(1-z)^3$. Hence,

$$\Re\left\{1+\frac{zf''(z)}{f'(z)}\right\} = \Re\left\{1+\frac{2z(1-z)^2}{(1-z)^3}\right\} = \Re\left\{\frac{1+z}{1-z}\right\} > 0.$$

Therefore, the function $z/(1-z)$ is convex in $\mathbb{D}$.

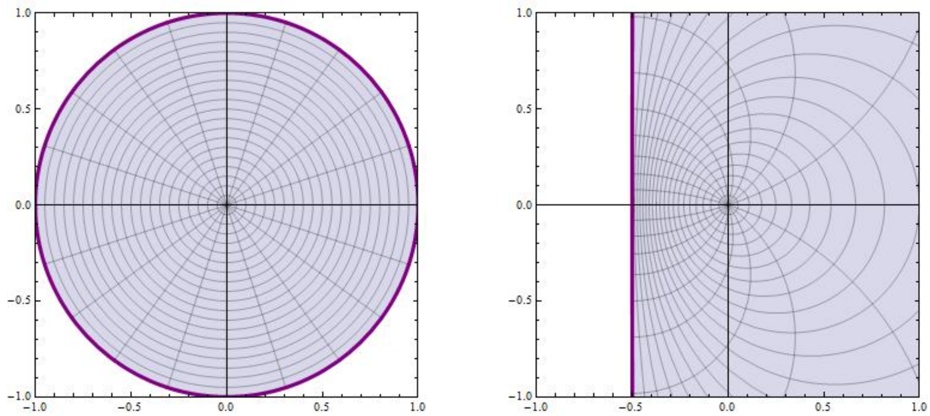

Figure 1.7: The image of unit disk $\mathbb{D}$ under the mapping $f(z) = z/(1-z)$. ■

Example 1.12. The function

$$f(z) = -\log(1-z) = \sum_{n=1}^{\infty} \frac{z^n}{n}$$

is analytic in $\mathbb{D}$ since f is differentiable at every $z \in \mathbb{D}$. Clearly, $f(0) = 0$. Since$f'(z) = 1/(1-z)$, it follows that $f'(0) = 1$. Also, $f''(z) = 1/(1-z)^2$. So,

$$\Re\left\{1+\frac{zf''(z)}{f'(z)}\right\} = \Re\left\{1+\frac{z(1-z)}{(1-z)^2}\right\} = \Re\left\{\frac{1}{1-z}\right\}.$$

From Example 1.4, it has been shown that $\Re\{1/(1-z)\} > 1/2$. It follows that $\Re\{1/(1-z)\} > 0$. Hence, the function $f(z) = -\log(1-z)$ is convex on $\mathbb{D}$.

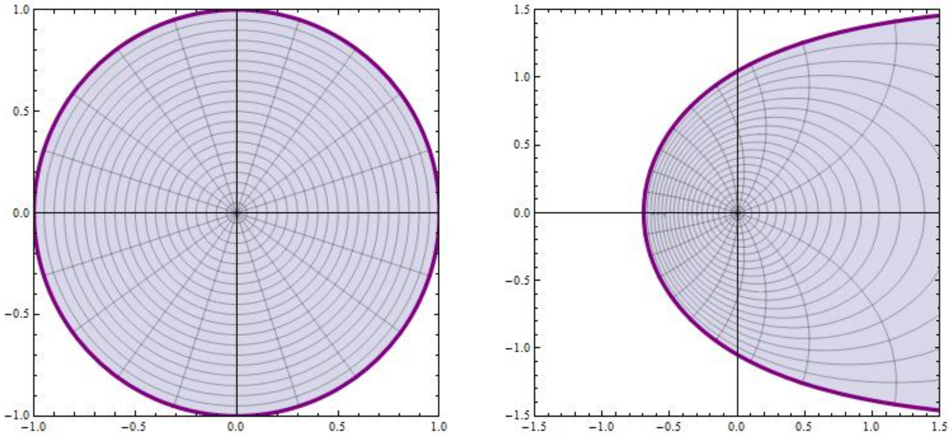

Figure 1.8: The image of $\mathbb{D}$ under the mapping $f(z) = -\log(1-z)$. ■

Example 1.13. The function

$$f(z) = \frac{1}{2}\log\frac{1+z}{1-z} = \sum_{n=0}^{\infty}\frac{z^{2n+1}}{2n+1}$$

is analytic in $\mathbb{D}$ since f is differentiable at every $z \in \mathbb{D}$. Clearly, $f(0) = 0$. Note that $f'(z) = 1/(1-z^2)$ and therefore $f'(0) = 1$. Also, $f''(z) = 2z/(1-z^2)^2$. Hence,

$$\Re\left\{1+\frac{zf''(z)}{f'(z)}\right\} = \Re\left\{1+\frac{2z^2(1-z^2)}{(1-z^2)^2}\right\} = \Re\left\{\frac{1+z^2}{1-z^2}\right\} > 0,$$

by Example 1.3. Therefore, $f(z) = (1/2)[\log(1+z)/(1-z)]$ is convex in $\mathbb{D}$.

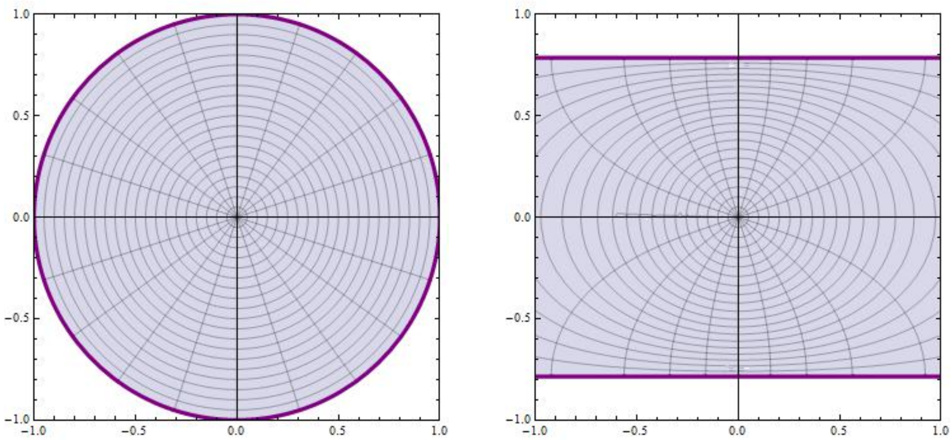

Figure 1.9: The image of $\mathbb{D}$ under the mapping $f(z) = (1/2)[\log(1+z)/(1-z)]$. ■

Remark 1.1. Every convex function f in $\mathbb{D}$ is evidently starlike because the convex domain $f(\mathbb{D})$ is also a starlike domain (starlike with respect to the origin) since f always maps origin to origin. The converse is not true in general as shown by the Koebe function, $k(z) = z/(1-z)^2$. We have seen in Example 1.8 that k is a starlike function. Now, note that since $k'(z) = (1+z)/(1-z)^3$ and $k''(z) = 2(z+2)/(1-z)^4$, it follows that

$$\Re\left\{1+\frac{zk''(z)}{k'(z)}\right\} = \Re\left\{1+\frac{2z(z+2)}{(1-z)^4}\frac{(1-z)^3}{(1+z)}\right\} = \Re\left\{\frac{z^2+4z+1}{1-z^2}\right\}.$$

For $z = -1/2 \in \mathbb{D}$, we have

$$\Re\left(\frac{z^2+4z+1}{1-z^2}\right) = -1 < 0.$$

Hence, Koebe function is not convex in $\mathbb{D}$. Alternatively, we can also show the Koebe function is not convex in geometric view. Recall that the Koebe function maps $\mathbb{D}$ maps $\mathbb{D}$ one-to-one and onto the entire complex plane minus the part of the negative axis from $-1/4$ to infinity. Consider the two points $-1/4+i$ and $-1/4-i$ in the image domain. Clearly, the line segment joining $-1/4+i$ and $-1/4-i$ does not lie inside the image domain. Therefore, the Koebe function is not convex in $\mathbb{D}$.

The two preceding theorems, that is, Theorem 1.6 and Theorem 1.7, provide a connection between starlikeness and convexity. This was first observed by Alexander [2] in 1915.

Theorem 1.8. (Alexander's Theorem) [2] A function $f \in \mathcal{A}$ is convex in $\mathbb{D}$ if and only if the function g defined by $g(z) = zf'(z)$ is starlike in $\mathbb{D}$.

Proof. If $g(z) = zf'(z)$, then

$$\frac{zg'(z)}{g(z)} = \frac{z(zf''(z) + f'(z))}{zf'(z)} = 1 + \frac{zf''(z)}{f'(z)}.$$

If the function f is convex, by Theorem 1.7, we have $\Re\{1 + zf''(z)/f'(z)\} > 0$. Since $\Re\{zg'(z)/g(z)\} = \Re\{1 + zf''(z)/f'(z)\} > 0$, the function g is starlike. The converse follows similarly from above. $\square$

The Alexander's Theorem (Theorem 1.8) can be rephrased in the form $f \in \mathcal{S}^*$ if and only if the function

$$g(z) = \int_0^z \frac{f(t)}{t}\, dt$$

is convex in $\mathbb{D}$.

Example 1.14. Consider the function $f(z) = z/(1-z)$. Since f is convex by Example 1.11, the function

$$g(z) = zf'(z) = \frac{z[(1-z) - z(-1)]}{(1-z)^2} = \frac{z}{(1-z)^2}$$

is starlike in $\mathbb{D}$. Notice that g is the Koebe function. $\blacksquare$

The Bieberbach conjecture for the class $\mathcal{S}^*$ of starlike functions holds true and it was proved by Nevalinna [35] in 1921.

Theorem 1.9. [35, see also 15] If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}^*$, then $|a_n| \leq n$ for all n. The inequality is sharp, as shown by the Koebe function, $k(z) = z/(1-z)^2$.

Using Alexander's Theorem (Theorem 1.8), the coefficient bound for class $\mathcal{C}$ of convex functions is easily deduced. This result was proved by Löewner [27] in 1917.

Theorem 1.10. [27, see also 15] If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{C}$, then $|a_n| \leq 1$ for all n. The inequality is sharp for all n.

Proof. Since $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is in $\mathcal{C}$, by Theorem 1.8,

$$zf'(z) = z + \sum_{n=2}^{\infty} n a_n z^n$$

is in $\mathcal{S}^*$. By Theorem 1.9, we have $n|a_n| \leq n$. Hence, $|a_n| \leq 1$. Since $z/(1-z) = z + z^2 + z^3 + \cdots$, and it is convex by Example 1.11, the bound $|a_n| \leq 1$ is sharp. $\square$

In 1936, Robertson [45] introduced the classes $\mathcal{S}^*(\alpha)$ and $\mathcal{C}(\alpha)$ of starlike and convex functions of order $\alpha, 0 \leq \alpha < 1$, respectively, which are defined as

$$\mathcal{S}^*(\alpha) = \left\{ f \in \mathcal{A} : \Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha \right\}$$

and

$$\mathcal{C}(\alpha) = \left\{ f \in \mathcal{A} : \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \right\}.$$

For $\alpha = 0$, we have $\mathcal{S}^*(0) := \mathcal{S}^*$ and $\mathcal{C}(0) := \mathcal{C}$. As α increases, both classes $\mathcal{S}^*(\alpha)$ and $\mathcal{C}(\alpha)$ become smaller. For $0 \leq \alpha < 1$, the geometrical interpretation of the notion of convexity of order α is that the ratio of the angle between two adjacent tangents to the unit circle to the angle between the two corresponding tangents of the image of the unit circle is less than $1/\alpha$ and comes arbitrarily close to $1/\alpha$ for some point of the unit circle [45]. Unfortunately, the class $\mathcal{S}^*(\alpha)$ do not admit any clear geometric interpretation for $0 \leq \alpha < 1$.

Example 1.15. Consider the function $k_\alpha(z) = z/(1-z)^{2(1-\alpha)}$, where $0 \leq \alpha < 1$. The function k_α is analytic in $\mathbb{D}$ since it is differentiable at all $z \in \mathbb{D}$. Clearly, $k_\alpha(0) = 0$. Since $k'_\alpha(z) = [1+(1-2\alpha)z]/(1-z)^{3-2\alpha}$, it follows that $k'_\alpha(0) = 1$. Note that

$$\Re\left\{\frac{zk'_\alpha(z)}{k_\alpha(z)}\right\} = \Re\left\{\frac{z(1+(1-2\alpha)z)}{(1-z)^{3-2\alpha}}\frac{(1-z)^{2(1-\alpha)}}{z}\right\} = \Re\left\{\frac{1+(1-2\alpha)z}{1-z}\right\} > \alpha.$$

Hence, k_α is in $\mathcal{S}^*(\alpha)$. This function k_α is called the Koebe function of order α, as $k_0(z) = z/(1-z)^2 = k(z)$, the Koebe function. ■

For $\alpha = 1/2$, we have the class of starlike functions of order $1/2$, that is,

$$\mathcal{S}^*(1/2) = \left\{f \in \mathcal{S} : \Re\left(\frac{zf'(z)}{f(z)}\right) > \frac{1}{2}\right\}.$$

Marx [30] and Strohhäcker [50] independently established the connection between the classes $\mathcal{C}$ and $\mathcal{S}^*(1/2)$.

Theorem 1.11. [30, 50] If $f \in \mathcal{C}$, then $f \in \mathcal{S}^*(1/2)$. This result is sharp, that is, the constant $1/2$ cannot be replaced by a larger constant.

Example 1.16. From Example 1.11, we know that the function $f(z) = z/(1-z)$ is convex. Hence, by Theorem 1.11, we can conclude that $f(z) = z/(1-z)$ is also starlike of order $1/2$. Alternatively, we can show directly that $\Re\{zf'(z)/f(z)\} > 1/2$. Note that $f'(z) = 1/(1-z)^2$. Hence

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} = \Re\left\{\frac{z}{(1-z)^2}\frac{(1-z)}{z}\right\} = \Re\left\{\frac{1}{1-z}\right\} > \frac{1}{2},$$

where the inequality follows from Example 1.4. ■

For $f \in \mathcal{S}^*(1/2)$, Schild [48] obtained the coefficient estimates as follows.

Theorem 1.12. [48] If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}^*(1/2)$, then $|a_n| \leq 1$. The inequality is sharp, as shown by the function $z/(1-z)$.

Another important subclass of univalent analytic functions is the class of close-to-convex functions, which was introduced by Kaplan [19].

Definition 1.6. [19] A function $f \in \mathcal{A}$ is said to be *close-to-convex* in $\mathbb{D}$ if there exists a convex function g in $\mathbb{D}$ such that

$$\Re\left\{\frac{f'(z)}{g'(z)}\right\} > 0,\ z \in \mathbb{D}. \tag{1.1}$$

We denote by $\mathcal{K}$ the class of close-to-convex functions in $\mathbb{D}$.

Every convex function is obviously close-to-convex in $\mathbb{D}$. Indeed, if f is convex in $\mathbb{D}$, then by choosing $g = f$ in (1.1), we have

$$\Re\left\{\frac{f'(z)}{g'(z)}\right\} = \Re\left\{\frac{f'(z)}{f'(z)}\right\} = 1 > 0.$$

Equivalently, the condition (1.1) can be written in the form

$$\Re\left\{\frac{zf'(z)}{h(z)}\right\} > 0,\ z \in \mathbb{D} \tag{1.2}$$

where $h(z) = zg'(z)$ is a starlike function on $\mathbb{D}$ by Alexander's Theorem (Theorem 1.8). In other words, a function $f \in \mathcal{A}$ is said to be *close-to-convex* in $\mathbb{D}$ if there exists a starlike function h such that the inequality (1.2) holds.

Suppose f is a starlike function in $\mathbb{D}$. If choose $h = f$ in (1.2), we have

$$\Re\left\{\frac{zf'(z)}{h(z)}\right\} = \Re\left\{\frac{zf'(z)}{f(z)}\right\} > 0.$$

Hence, we can conclude that every starlike function is close-to-convex in $\mathbb{D}$.

Therefore, we have the following inclusion

$$\mathcal{C} \subset \mathcal{S}^* \subset \mathcal{K}.$$

From this, instant examples of close-to-convex functions are $z/(1-z)$ and the Koebe functions, $k(z) = z/(1-z)^2$. Now, it is also natural to ask if close-to-convex functions are univalent. Kaplan [19] showed that they are indeed so.

Theorem 1.13. [19] Every close-to-convex function is univalent.

Proof. Suppose f is close-to-convex in $\mathbb{D}$. By Definition 1.6, there exists a convex function g in $\mathbb{D}$ in such that $\Re\{f'(z)/g'(z)\} > 0$. Since g is convex, it follows that g maps $\mathbb{D}$ one-to-one and onto convex domain $g(\mathbb{D})$. Therefore, g^{-1} exists in $g(\mathbb{D})$. Consider the function

$$h(w) = f(g^{-1}(w)),\ w \in g(\mathbb{D}). \tag{1.3}$$

Since g is analytic $\mathbb{D}$, it follows that g^{-1} is also analytic in $g(\mathbb{D})$. Using the fact that the composition of two analytic functions is analytic, the function h is analytic in $\mathbb{D}$.

Differentiating (1.3), we obtain

$$h'(w) = \frac{f'(g^{-1}(w))}{g'(g^{-1}(w))} = \frac{f'(z)}{g'(z)}, \quad w \in g(\mathbb{D}),\, z \in \mathbb{D},$$

and $\Re\{h'(w)\} = \Re\{f'(z)/g'(z)\} > 0$ in $g(\mathbb{D})$. Now, by the Noshiro-Warschawski Theoreom (Theorem 1.1), the function h is univalent in $g(\mathbb{D})$. Therefore, equation (1.3) becomes

$$f(z) = h(g(z)), \quad z \in \mathbb{D}.$$

Using the fact that the composition of two univalent functions is again univalent, the function f is univalent in $\mathbb{D}$. $\square$

With Theorem 1.13, we have the inclusion

$$\mathcal{C} \subset \mathcal{S}^* \subset \mathcal{K} \subset \mathcal{S}.$$

Remark 1.2. Recall that every starlike function in $\mathbb{D}$ is close-to-convex but the converse is not necessarily true. We will show an example of a close-to-convex function which is not starlike. Such a function is

$$f(z) = z + \frac{1}{2}z^2 + \frac{1}{3}z^3. \tag{1.4}$$

To show that this f is close-to-convex, we need a lemma due to Ozaki [38].

Lemma 1.1. [38] If $f(z) = z + \sum_{n\geq 2} A_n z^n$ is analytic in $\mathbb{D}$ and if $1 \geq 2A_2 \geq \cdots \geq nA_n \geq \cdots \geq 0$ or $1 \leq 2A_2 \leq \cdots \leq nA_n \leq \cdots \leq 2$, then f is close-to-convex with respect to $-\log(1-z)$.

Observe that the second coefficient A_2 and third coefficient A_3 of the function f in (1.4) are $1/2$ and $1/3$, respectively. Also, f satisfies the hypothesis $1 \geq 2A_2 \geq 3A_3 \geq \cdots \geq 0$. Therefore, f is close-to-convex with respect to $-\log(1-z)$.

To show that f is not a starlike function, note that for $z = e^{i\theta}$, we have

$$\begin{aligned}
\Re\left\{\frac{f(z)}{zf'(z)}\right\} &= \Re\left\{\frac{1+(1/2)z+(1/3)z^2}{1+z+z^2}\right\} \\
&= \Re\left\{\frac{1+(1/2)e^{i\theta}+(1/3)e^{2i\theta}}{1+e^{i\theta}+e^{2i\theta}}\right\} \\
&= \Re\left\{\frac{6e^{-i\theta}+3+2e^{i\theta}}{6(e^{-i\theta}+1+e^{i\theta})}\right\} \\
&= \Re\left\{\frac{6\cos\theta-6i\sin\theta+3+2\cos\theta+2i\sin\theta}{6(\cos\theta-i\sin\theta+1+\cos\theta+i\sin\theta)},\right\} \\
&= \frac{3+8\cos\theta}{6(1+2\cos\theta)},
\end{aligned}$$

which is negative for some values of θ, for example, when $\cos\theta = -2/5$. For a function $p(z)$, which is nonzero, we have

$$\Re\left\{\frac{1}{p(z)}\right\} = \Re\left\{\frac{\overline{p(z)}}{|p(z)|^2}\right\} = \frac{1}{|p(z)|^2}\Re\{p(z)\}.$$

If $\Re\{1/p(z)\} < 0$, then $\Re\{p(z)\} < 0$. Since $\Re\{f(z)/zf'(z)\} < 0$ for $z = e^{i\theta_0}$, $\cos\theta_0 = -2/5$, it follows that $\Re\{zf'(z)/f(z)\} < 0$ there. Thus, f is not starlike in $\mathbb{D}$. Hence, a close-to-convex function is not necessarily starlike.

Recall that the functions $z, -\log(1-z), (1/2)\log[(1+z)/(1-z)]$ and $z/(1-z)$ are convex functions in $\mathbb{D}$ (Examples 1.10 - 1.13). Hence, by choosing suitable convex functions in (1.1), we obtain some sufficient conditions for a function to be close-to-convex.

Theorem 1.14. If $f \in \mathcal{A}$ satisfying any one of the following conditions

$$\Re\{f'(z)\} > 0, \tag{1.5}$$

$$\Re\{(1-z)f'(z)\} > 0, \tag{1.6}$$

$$\Re\{(1-z^2)f'(z)\} > 0, \tag{1.7}$$

$$\Re\{(1-z)^2 f'(z)\} > 0, \tag{1.8}$$

in $\mathbb{D}$, then f is in $\mathcal{K}$.

Proof. To obtain (1.5) - (1.8), we choose in Definition 1.6 the convex function g respectively to be $z, -\log(1-z), \frac{1}{2}\log[(1+z)/(1-z)]$ and $z/(1-z)$. □

Recall that every close-to-convex function is univalent in $\mathbb{D}$, hence is locally univalent. However, the converse may not be true. The following theorem gives a necessary and sufficient condition for a locally univalent analytic function to be close-to-convex in $\mathbb{D}$.

Theorem 1.15. [19] (Kaplan's Theorem) Let f be analytic and $f'(z) \neq 0$ in $\mathbb{D}$. Then f is close-to-convex if and only if

$$\int_{\theta_1}^{\theta_2} \Re\left[1 + \frac{re^{i\theta} f''(re^{i\theta})}{f'(re^{i\theta})}\right] d\theta > -\pi,$$

for each $r \in (0,1)$ and for each pair of real numbers θ_1, θ_2 such that $0 \leq \theta_2 - \theta_1 \leq 2\pi$.

Geometrically, Kaplan's theorem implies that the image of each circle $|z| = r < 1$ is a simple closed curve with the property that as θ increases, either in the counterclockwise direction or clockwise direction, the angle of the tangent vector $\arg\{(\partial/\partial\theta)f(re^{i\theta})\}$ does not decrease by more than $-\pi$ in any interval $[\theta_1, \theta_2]$. In other words, the curve cannot make a "hairpin bend" backward to intersect itself. (Refer Kaplan [19] page 177.)

The Bieberbach conjecture also holds for close-to-convex functions as was proved by Reade [44] in 1955.

Theorem 1.16. [44] If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is close-to-convex, then $|a_n| \leq n$, for all n. The inequality is sharp, as shown by the Koebe function, $k(z) = z/(1-z)$.

Proof. Suppose f is close-to-convex. By Definition 1.6, there exists a convex function g such that $\Re\{f'(z)/g'(z)\} > 0$. Let $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$. Since g is convex, by Theorem 1.10, we know that $|b_n| \leq 1$ for all n. Since $f'(z)/g'(z) \in \mathcal{P}$, it follows from Theorem 1.2 that the series representation of $f'(z)/g'(z)$ is given by

$$\frac{f'(z)}{g'(z)} = 1 + \sum_{n=1}^{\infty} c_n z^n. \tag{1.9}$$

By Theorem 1.4, we have $|c_n| \leq 2$. Since

$$\frac{f'(z)}{g'(z)} = \frac{1 + \sum_{n=2}^{\infty} n a_n z^{n-1}}{1 + \sum_{n=2}^{\infty} n b_n z^{n-1}},$$

we have

$$
\begin{aligned}
1+\sum_{n=2}^{\infty} na_n z^{n-1} &= \left(1+\sum_{n=2}^{\infty} nb_n z^{n-1}\right)\left(1+\sum_{n=1}^{\infty} c_n z^n\right) \\
&= 1+\sum_{n=2}^{\infty} nb_n z^{n-1}+\sum_{n=1}^{\infty} c_n z^n+\left(\sum_{n=2}^{\infty} nb_n z^{n-1}\right)\left(\sum_{n=1}^{\infty} c_n z^n\right) \\
&= 1+\sum_{n=2}^{\infty} nb_n z^{n-1}+\sum_{n=2}^{\infty} c_{n-1} z^{n-1}+\left(\sum_{n=2}^{\infty} nb_n z^{n-1}\right)\left(\sum_{n=2}^{\infty} c_{n-1} z^{n-1}\right).
\end{aligned}
$$

Equating the coefficient of z^{n-1} on both sides, we get

$$
na_n = nb_n+(n-1)b_{n-1}c_1+(n-2)b_{n-2}c_2+\cdots+2b_2c_{n-2}+c_{n-1},
$$

which yields

$$
|na_n| \leq n|b_n|+(n-1)|b_{n-1}||c_1|+(n-2)|b_{n-2}||c_2|+\cdots+2|b_2||c_{n-2}|+|c_{n-1}|.
$$

Since $|b_n| \leq 1$ and each $|c_n| \leq 2$ for any $n \in \mathbb{N}$, we have

$$
\begin{aligned}
|na_n| &\leq n+2(n-1)+2(n-2)\cdots+4+2 \\
&= n+2[(n-1)+(n-2)+\cdots+2+1] \\
&= n+2\left[\frac{n(n-1)}{2}\right] \\
&= n^2,
\end{aligned}
$$

which implies that $|a_n| \leq n$ for all $n = 1,2,3,\ldots$. $\square$

1.3 Scope Of Dissertation

Here is the summary of the dissertation. The dissertation is divided into four chapters, followed by references at the end.

In the first chapter, which is the introductory chapter, we review and assemble some of the general principles of theory of univalent functions which underlie the geometric function theory of a complex variable.

Chapter 2 deals with sufficient conditions for analytic functions satisfying certain third-order differential inequalities to be starlike in the unit disk $\mathbb{D}$. As a consequence, conditions for starlikeness of functions defined by integral operators are obtained. Connections are also made to earlier known results. Some background on differential inequalities and operator theory are discussed.

Chapter 3 studies a new subclass of close-to-convex functions. Some interesting results are obtained such as inclusion relationships, an estimate for the Fekete-Szegö functional for functions belonging to the class, coefficient estimates, and a sufficient condition. Connections are made with previously known results

In Chapter 4, a summary of the work done in this dissertation is presented.

CHAPTER 2

STARLIKENESS OF AN INTEGRAL OPERATOR

2.1 Introduction

In real analysis, we all learn that if the first derivative of a function f is positive, that is, $f'(x) > 0$, then f is an increasing function. In complex analysis, if f is analytic in a convex domain and the real part of its first derivative is positive, that is, $\Re\{f'(z)\} > 0$, then f is univalent in the domain. This is the well-known Noshiro-Warschawski Theorem discussed earlier in Theorem 1.1.

An important area of research in geometric function theory is to determine sufficient conditions to ensure starlikeness of analytic functions in the unit disk $\mathbb{D}$. These include conditions in terms of differential inequalities. One of the famous differential inequalities is the Alexander differential $g(z) = zf'(z)$, which provides a connection between starlikeness and convexity of analytic functions, as discussed in Theorem 1.8. Moreover, many researchers also investigate the starlikeness of analytic functions given by operators. The study of operators also plays a vital role in complex function theory. In the literature, there are several well known operators such as the Alexander integral operator, Libera integral operator and Bernardi integral operator.

Early in 1915, Alexander [2] introduced the operator $A : \mathcal{A} \to \mathcal{A}$ defined by

$$A[f](z) := \int_0^z \frac{f(t)}{t}\, dt.$$

This operator is now known as the Alexander operator. This operator establishes a connection between starlikeness and convexity of analytic functions.

Meanwhile, Libera [26] studied another operator $L : \mathcal{A} \to \mathcal{A}$ given by

$$g(z) = L[f](z) = \frac{2}{z}\int_0^z f(t)\,dt, \quad f \in \mathcal{A}, \tag{2.1}$$

which is knowns as the Libera operator. The Libera integral is also the solution of the first-order linear differential equation: $zg'(z) + g(z) = 2f(z)$. He also showed that the operator preserved starlikeness, convexity and close-to-convexity.

Theorem 2.1. [26, Theorem 1 and Theorem 2] If f is in $\mathcal{S}^*$, then the function defined by (2.1) is likewise in $\mathcal{S}^*$. This result is also holds true for $f \in \mathcal{C}$.

Theorem 2.2. [26, Theorem 3] If f is close-to-convex with respect to g, then $L[f]$ is also close-to-convex with respect to $L[g]$ where $L[f]$ is defined in (2.1) and

$$L[g](z) = \frac{2}{z}\int_0^z g(t)\,dt.$$

In 1969, Bernardi [3] introduced an operator $L_\gamma : \mathcal{A} \to \mathcal{A}$ defined by

$$h(z) = L_\gamma[f](z) = \frac{1+\gamma}{z^\gamma}\int_0^z f(t)t^{\gamma-1}\,dt, \quad \gamma = 0, 1, 2, 3, \ldots. \tag{2.2}$$

This operator is known as the Bernardi operator. For $\gamma = 0$, we have

$$L_0[f](z) := \int_0^z \frac{f(t)}{t}\,dt = A[f](z)$$

which is the Alexander operator. For $\gamma = 1$, it can be easily seen that

$$L_1[f](z) := \frac{2}{z}\int_0^z f(t)\,dt = L[f](z).$$

Hence, the Bernardi operator is a generalization of the Libera operator. The Bernardi integral is the solution of the differential equation: $zh'(z) + \gamma h(z) = (1+\gamma)f(z)$.

Pascu [40] and Lewandowski *et al.*[25] independently showed that the Bernardi operator also preserves starlikeness, convexity and close-to-convexity, even when γ is a complex number.

Theorem 2.3. [25, 40] Let L_γ be defined in (2.2) and $\Re\{\gamma\} \geq 0$. Then

(i) $L_\gamma[\mathcal{S}^*] = \{L_\gamma[f](z) \mid f \in \mathcal{S}^*\} \subset \mathcal{S}^*$,

(ii) $L_\gamma[\mathcal{C}] = \{L_\gamma[f](z) \mid f \in \mathcal{C}\} \subset \mathcal{C}$,

(iii) $L_\gamma[\mathcal{K}] = \{L_\gamma[f](z) \mid f \in \mathcal{K}\} \subset \mathcal{K}$.

On the other hand, Miller *et al.* [32] gave the definition of a starlike operator. An operator defined on $\mathcal{S}^*$, that maps $\mathcal{S}^*$ into (or onto) $\mathcal{S}^*$, is called a *starlike operator*. Hence, the Alexander, Libera and Bernardi operators are examples of starlike operators.

2.2 Motivation

Recall that

$$\mathcal{A}_n = \left\{ f \in \mathcal{H} : f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k \right\},$$

where $\mathcal{H}$ denote the class of all analytic functions in $\mathbb{D}$ and $\mathcal{A}_1 := \mathcal{A}$. For $f \in \mathcal{A}_n$, Mocanu [33] considered the problem of finding the maximum value of λ for which $|f''(z)| \leq \lambda$ implies f is starlike in $\mathbb{D}$ and proved the following result.

Theorem 2.4. [33, Theorem 2] If $f \in \mathcal{A}_n$ and

$$|f''(z)| \leq \frac{n(n+1)}{2n+1},$$

then $f \in \mathcal{S}^*$.

In 1993, Mocanu [34] improved the bound of Theorem 2.4 as follows.

Theorem 2.5. [34, Theorem] If $f \in \mathcal{A}_n$ and

$$|f''(z)| \leq \frac{n(n+1)}{\sqrt{(n+1)^2+1}},$$

then $f \in \mathcal{S}^*$.

Remark 2.1. Putting $n = 1$ in Theorem 2.5, whatever $f \in \mathcal{A} := A_1$ and $|f''(z)| \leq 2/\sqrt{5}$, then $f \in \mathcal{S}^*$.

Finally, Obradović [37] closed this problem with the constant $\lambda = 1$ by proving that this result is sharp for $f \in \mathcal{A}$.

Theorem 2.6. [37, Theorem 1] If $f \in \mathcal{A}$ and $|f''(z)| \leq 1$, $z \in \mathbb{D}$, then $f \in \mathcal{S}^*$. The result is sharp.

In 2003, Fournier and Mocanu [12] studied some second order differential inequalities which imply starlikeness.

Theorem 2.7. [12, Theorem 2] Let $f \in \mathcal{A}$ and $0 \leq \alpha < 1$. If

$$|zf''(z) - \alpha(f'(z) - 1)| \leq 1 - \alpha,$$

then $f \in \mathcal{S}^*$.

Later, Miller and Mocanu [31] generalized Theorem 2.7 by considering $f \in \mathcal{A}_n$.

Theorem 2.8. [31, Lemma 2.1] Let $f \in \mathcal{A}_n$ and $0 \leq \alpha < n$. If

$$|zf''(z) - \alpha(f'(z) - 1)| < n - \alpha,$$

then $f \in \mathcal{S}^*$.

Besides, they also obtained another second-order differential inequality that provides a condition for starlikeness.

Theorem 2.9. [31, Lemma 2.2] Let $f \in \mathcal{A}_n$ and $0 \leq \alpha < n+1$. If

$$\left| zf''(z) - \alpha\left(f'(z) - \frac{f(z)}{z}\right)\right| < \frac{n(n+1-\alpha)}{n+1},$$

then $f \in \mathcal{S}^*$.

Later, Kuroki and Owa [24] extended Theorem 2.8 to obtain a condition for starlikeness of order β.

Theorem 2.10. [24, Theorem 2.1] Let $f \in \mathcal{A}_n, 0 \leq \alpha < n$ and $0 \leq \beta < 1$. If

$$|zf''(z) - \alpha(f'(z) - 1)| < \frac{(n+1)(1-\beta)(n-\alpha)}{n+1-\beta},$$

then $f \in \mathcal{S}^*(\beta)$.

The extension of Theorem 2.9 for $f \in \mathcal{A}_n$ to be starlike of order β was done by Verma *et al.*[52].

Theorem 2.11. [52, Theorem 3.1] Let $f \in \mathcal{A}_n$ and α and β be real numbers such that $0 \leq \alpha < n+1$ and $0 \leq \beta < 1$. If

$$\left| zf''(z) - \alpha\left(f'(z) - \frac{f(z)}{z}\right)\right| < \frac{n(n+1-\alpha)(1-\beta)}{(n+1-\beta)},$$

then $f \in \mathcal{S}^*(\beta)$.

Furthermore, Aghalary and Joshi [1] extended Theorem 2.10 by considering α to be a complex number.

Theorem 2.12. [1, Theorem 2.3] Let $f \in \mathcal{A}_n, 0 \leq \beta < 1$ and $\max\{0, |\alpha| + \beta - 1\} \leq \Re\{\alpha\} < n$ where $\alpha \in \mathbb{C}$. If f satisfies

$$|zf''(z) - \alpha(f'(z) - 1)| < \frac{(n - \Re\,\alpha)(n+1)[1 + \Re\,\alpha - |\alpha| - \beta]}{n+1-\beta},$$

then $f \in \mathcal{S}^*(\beta)$.

Recently, Chandrashekar *et al.*[6] obtained sufficient condition on certain third-order differential inequality that would imply starlikeness of an analytic function. The results obtained extend the result of Kuroki and Owa [24] (Theorem 2.10).

Theorem 2.13. [6, Theorem 6] Let $f \in \mathcal{A}_n, 0 < \alpha < n\nu, \delta > \alpha \geq \gamma \geq 0$, and $0 \leq \beta < 1$.

Further let μ and ν satisfy

$$\nu - \alpha\mu = \delta - \gamma \quad \text{and} \quad \nu\mu = \gamma.$$

If

$$|\gamma z^2 f'''(z) + \delta z f''(z) - \alpha(f'(z) - 1)| < \frac{(1+n\mu)(n+1)(1-\beta)(n\nu - \alpha)}{n+1-\beta},$$

then $f \in \mathcal{S}^*(\beta)$.

Recall that the Alexander operator is defined by

$$A[f](z) := \int_0^z \frac{f(t)}{t}\, dt,$$

where $A : \mathcal{A} \to \mathcal{A}$. It can seen that

$$\int_0^z \frac{f(t)}{t}\, dt = \int_0^1 \frac{f(rz)}{r}\, dr,$$

by setting $t = rz$. For single operators of the form

$$f(z) = \int_0^1 W(r,z)\, dr,$$

Miller *et al.*[32] have determined conditions on the kernel W to ensure f to be a starlike function by using the theory of differential subordination. Aghalary and Joshi [1], Fournier and Mocanu [12], Kuroki and Owa [24], Miller and Mocanu [31] and Verma *et al.*[51] determined conditions for starlikeness of functions defined by double integral operator

$$f(z)=\int_0^1\int_0^1 W(r,s,z)\,dr\,ds.$$

More recently, Chandrashekar *et al.*[6] obtanied conditions for starlikeness of functions defined by triple integral operators

$$f(z)=\int_0^1\int_0^1\int_0^1 W(r,s,t,z)\,dr\,ds\,dt.$$

Using Theorem 2.8, Miller and Mocanu [31] obtained result concerning the double integral starlike operator.

Theorem 2.14. [31, Theorem 2.1] Let $0\leq\alpha<n$ and $g\in\mathcal{H}$. If $|g(z)|\leq n-\alpha$, then

$$f(z)=z+z^{n+1}\int_0^1\int_0^1 g(rsz)r^{n-\alpha-1}s^n\,dr\,ds$$

is a starlike function.

Using Theorem 2.13, Chandrashekar *et al.*[6] constructed a starlike function of order β expressed in terms of a triple integral.

Theorem 2.15. [6, Theorem 7] Let $0<\alpha<n\mu,\ \delta>\alpha\geq\gamma>0,\ 0\leq\beta<1$, and $g\in\mathcal{H}$. If

$$|g(z)|<\frac{(1+n\mu)(n+1)(1-\beta)(n\nu-\alpha)}{n+1-\beta}$$

where

$$\nu-\alpha\mu=\delta-\gamma, \nu\mu=\gamma,$$

then

$$f(z) = z + \frac{z^{n+1}}{\gamma} \int_0^1 \int_0^1 \int_0^1 g(rstz) r^{n-1-\alpha/\nu} s^n t^{n-1+1/\mu} drdsdt$$

satisfies $f \in \mathcal{S}^*(\beta)$.

Motivated by aforementioned works, the aim of the present work is to obtain another third order differential inequality which gives a sufficient condition for functions in $\mathcal{A}_n$ to be starlike functions of order β. Using this third-order differential inequality, we construct new starlike function of order β which can be expressed in terms of triple integral of some function in the class $\mathcal{H}$.

To prove the main results, the following lemmas will be used.

Lemma 2.1. [17] Let h be convex in $\mathbb{D}$ with $h(0) = a$ and $\Re\{\gamma\} \geq 0$. If $p(z) \in \mathcal{H}[a,n]$ and

$$p(z) + \frac{zp'(z)}{\gamma} \prec h(z),$$

then

$$p(z) \prec q(z) \prec h(z),$$

where

$$q(z) = \frac{\gamma}{nz^{\gamma/n}} \int_0^z h(t) t^{(\gamma/n)-1} \, dt.$$

The function q is convex. This result is sharp, that is, the function q is the best $(a,n)-$ dominant.

Lemma 2.2. [51] Suppose μ, ν are real numbers and satisfy

$$\mu + \nu = \alpha - \gamma \quad \text{and} \quad \mu\nu = \gamma$$

such that $\mu > 0$ and $\nu > 2/(1-\beta)$ where $0 \leq \beta < 1$. If $f \in \mathcal{A}_n$ satisfies

$$\left|(1-\alpha+2\gamma)\frac{f(z)}{z} + (\alpha-2\gamma)f'(z) + \gamma z f''(z) - 1\right| < \frac{(1+n\mu)(1+n\nu)(\nu(1-\beta)-2)}{\nu(n+1-\beta)}$$

for $z \in \mathbb{D}$, then $f \in \mathcal{S}^*(\beta)$.

Lemma 2.3. [51] Suppose $g \in \mathcal{H}$ satisfies

$$|g(z)| \leq \frac{(1+n\mu)(1+n\nu)(\nu(1-\beta)-2)}{\nu(n+1-\beta)}$$

for some $\mu > 0, \nu > 2/(1-\beta)$ and $0 \leq \beta < 1$. Then the function f given by

$$f(z) = z + \frac{z^{n+1}}{\mu\nu}\int_0^1\int_0^1 g(rsz)r^{n+1/\mu-1}s^{n+1/\nu-1}drds$$

is starlike of order β in $\mathbb{D}$.

2.3 Main results

Theorem 2.16. Let $f \in \mathcal{A}_n$, $\delta \geq 0, \mu > 0$, $\nu > 2/(1-\beta)$ and $0 \leq \beta < 1$. Further let μ and ν satisfy

$$\mu + \nu = \alpha - \gamma \quad \text{and} \quad \mu\nu = \gamma.$$

If

$$\begin{aligned}\Big|\delta\gamma z^2 f'''(z) + [\gamma + \delta(\alpha-\gamma)]zf''(z) + [\delta(1-\alpha+2\gamma) + (\alpha-2\gamma)]f'(z) + \\ (1-\alpha+2\gamma)(1-\delta)\frac{f(z)}{z} - 1\Big| \\ < \frac{(1+n\delta)(1+n\mu)(1+n\nu)[\nu(1-\beta)-2]}{\nu(n+1-\beta)},\end{aligned} \quad (2.3)$$

then $f \in \mathcal{S}^*(\beta)$.

Remark 2.2. Putting $\delta = 0$ in Theorem 2.16, then $f \in \mathcal{S}^*(\beta)$ if

$$\left|(1-\alpha+2\gamma)\frac{f(z)}{z}+(\alpha-2\gamma)f'(z)+\gamma z f''(z)-1\right| < \frac{(1+n\mu)(1+n\nu)(\nu(1-\beta)-2)}{\nu(n+1-\beta)}.$$

This result was obtained earlier by Verma *et al.*[51, Theorem 3.1].

Proof. The differential inequality (2.3) can be written as follows:

$$\begin{aligned}\delta\gamma z^2 f'''(z)+[\gamma+\delta(\alpha-\gamma)]zf''(z)+[\delta(1-\alpha+2\gamma)+(\alpha-2\gamma)]f'(z)+&\\ (1-\alpha+2\gamma)(1-\delta)\frac{f(z)}{z}&\\ \prec 1+\frac{(1+n\delta)(1+n\mu)(1+n\nu)[\nu(1-\beta)-2]}{\nu(n+1-\beta)}z.&\end{aligned} \quad (2.4)$$

Let

$$p(z) = (1-\alpha+2\gamma)\frac{f(z)}{z}+(\alpha-2\gamma)f'(z)+\gamma z f''(z)-1.$$

Then,

$$\begin{aligned}p(z)+\delta z p'(z) &= (1-\alpha+2\gamma)\frac{f(z)}{z}+(\alpha-2\gamma)f'(z)+\gamma z f''(z)-1\\ &\quad+\delta z\Big((1-\alpha+2\gamma)\frac{zf'(z)-f(z)}{z^2}+(\alpha-2\gamma)f''(z)+\gamma(zf'''(z)+f''(z))\Big)\\ &= (1-\alpha+2\gamma)\frac{f(z)}{z}+(\alpha-2\gamma)f'(z)+\gamma z f''(z)-1+\delta(1-\alpha+2\gamma)f'(z)\\ &\quad-\delta(1-\alpha+2\gamma)\frac{f(z)}{z}+\delta(\alpha-2\gamma)zf''(z)+\delta\gamma z^2 f'''(z)+\delta\gamma z f''(z)\\ &= \delta\gamma z^2 f'''(z)+\big(\gamma+\delta(\alpha-\gamma)\big)zf''(z)+\big(\delta(1-\alpha+2\gamma)+(\alpha-2\gamma)\big)f'(z)\\ &\quad+(1-\alpha+2\gamma)(1-\delta)\frac{f(z)}{z}-1.\end{aligned}$$

Hence, (2.4) can be written as

$$p(z)+\delta zp'(z)\prec\frac{(1+n\delta)(1+n\mu)(1+n\nu)(\nu(1-\beta)-2)}{\nu(n+1-\beta)}z=h(z).$$

Since the function h satisfying the inequality

$$\Re\left\{1+\frac{zh''(z)}{h'(z)}\right\}=1>0,$$

it follows that the function h is convex. Also, $h(0)=p(0)=0$. It follows from Lemma 2.1 with $\gamma=1/\delta$ that

$$\begin{aligned}p(z)&\prec\frac{1}{\delta nz^{1/\delta n}}\int_0^z\frac{(1+n\delta)(1+n\mu)(1+n\nu)(\nu(1-\beta)-2)}{\nu(n+1-\beta)}t\cdot t^{1/\delta n-1}dt\\&\prec\frac{1}{\delta nz^{1/\delta n}}\frac{(1+n\delta)(1+n\mu)(1+n\nu)(\nu(1-\beta)-2)}{\nu(n+1-\beta)}\int_0^z t^{1/\delta n}dt\\&\prec\frac{(1+n\mu)(1+n\nu)(\nu(1-\beta)-2)}{\nu(n+1-\beta)}z,\end{aligned}$$

which gives

$$(1-\alpha+2\gamma)\frac{f(z)}{z}+(\alpha-2\gamma)f'(z)+\gamma zf''(z)-1\prec\frac{(1+n\mu)(1+n\nu)(\nu(1-\beta)-2)}{\nu(n+1-\beta)}z.$$

Hence, we have

$$(1-\alpha+2\gamma)\frac{f(z)}{z}+(\alpha-2\gamma)f'(z)+\gamma zf''(z)-1=\frac{(1+n\mu)(1+n\nu)(\nu(1-\beta)-2)}{\nu(n+1-\beta)}\omega(z).$$

where $\omega(z)$ is a Schwarz function as defined in Definition 1.4. Taking modulus on both sides, we obtain

$$\left|(1-\alpha+2\gamma)\frac{f(z)}{z}+(\alpha-2\gamma)f'(z)+\gamma z f''(z)-1\right| = \left|\frac{(1+n\mu)(1+n\nu)(\nu(1-\beta)-2)}{\nu(n+1-\beta)}\right||\omega(z)|$$
$$< \frac{(1+n\mu)(1+n\nu)(\nu(1-\beta)-2)}{\nu(n+1-\beta).}$$

Hence, by Lemma 2.2, $f \in \mathcal{S}^*(\beta)$. $\square$

Using Theorem 2.16, the conditions on the starlikeness of a triple integral is obtained.

Theorem 2.17. Let $g \in \mathcal{H}$, $\delta > 0, \mu > 0$, $\nu > 2/(1-\beta)$ and $0 \leq \beta < 1$. If

$$|g(z)| < \frac{(1+n\delta)(1+n\mu)(1+n\nu)(\nu(1-\beta)-2)}{\nu(n+1-\beta)}, \tag{2.5}$$

where

$$\mu+\nu = \alpha-\gamma \quad \text{and} \quad \mu\nu = \gamma$$

then

$$f(z) = z + \frac{z^{n+1}}{\delta\gamma}\int_0^1\int_0^1\int_0^1 g(rstz) r^{n+(1/\mu)-1} s^{n+(1/\nu)-1} t^{n+(1/\delta)-1}\, dr\, ds\, dt$$

is in the class $\mathcal{S}^*(\beta)$.

Proof. Consider

$$f(z) = z + \frac{z^{n+1}}{\delta\gamma}\int_0^1\int_0^1\int_0^1 g(rstz) r^{n+(1/\mu)-1} s^{n+(1/\nu)-1} t^{n+(1/\delta)-1}\, dr\, ds\, dt. \tag{2.6}$$

The function f is analytic in $\mathbb{D}$ and has the form $f(z)=z+\sum_{k=n+1}^{\infty}a_kz^k$. Hence, $f\in\mathcal{A}_n$.

Setting $\phi(rsz)=\frac{1}{\delta}\int_0^1 g(rstz)t^{n+(1/\delta)-1}\,dt$, equation (2.6) becomes

$$\begin{aligned}
f(z) &= z+\frac{z^{n+1}}{\gamma}\int_0^1\int_0^1 \phi(rsz)r^{n+(1/\mu)-1}s^{n+(1/\nu)-1}\,dr\,ds\\
&= z+\frac{z^{n+1}}{\mu\nu}\int_0^1\left(\int_0^1 \phi(rsz)r^{n+(1/\mu)-1}\,dr\right)s^{n+(1/\nu)-1}\,ds. \qquad (2.7)
\end{aligned}$$

Let $\zeta=sz$. Then equation (2.7) becomes

$$\begin{aligned}
f(z) &= z+\frac{z^{n+1}}{\mu\nu}\int_0^z\left(\int_0^1 \phi(r\zeta)r^{n+(1/\mu)-1}\,dr\right)(\zeta z^{-1})^{n+(1/\nu)-1}z^{-1}d\zeta\\
&= z+\frac{z^{1-1/\nu}}{\mu\nu}\int_0^z\left(\int_0^1 \phi(r\zeta)r^{n+(1/\mu)-1}\,dr\right)\zeta^{n+(1/\nu)-1}\,d\zeta\\
\frac{f(z)}{z} &= 1+\frac{1}{\mu\nu z^{1/\nu}}\int_0^z\left(\int_0^1 \phi(r\zeta)r^{n+(1/\mu)-1}\,dr\right)\zeta^{n+(1/\nu)-1}\,d\zeta.
\end{aligned}$$

By setting $\varphi(z)=\frac{f(z)}{z}$, we can get

$$z^{1/\nu}\varphi(z)=z^{1/\nu}+\frac{1}{\mu\nu}\int_0^z\left(\int_0^1 \phi(r\zeta)r^{n+(1/\mu)-1}\,dr\right)\zeta^{n+(1/\nu)-1}\,d\zeta.$$

Differentiating both sides, we have

$$z^{1/\nu}\varphi'(z)+\frac{z^{(1/\nu)-1}\varphi(z)}{\nu}=\frac{z^{(1/\nu)-1}}{\nu}+\frac{z^{n+(1/\nu)-1}}{\mu\nu}\int_0^1 \phi(rz)r^{n+(1/\mu)-1}dr$$

or

$$\nu z\varphi'(z)+\varphi(z)=1+\frac{z^n}{\mu}\int_0^1 \phi(rz)r^{n+(1/\mu)-1}dr.$$

Since $\varphi'(z) = (f'(z) - f(z))/z$, it follows that

$$(1-\nu)\frac{f(z)}{z} + \nu f'(z) = 1 + \frac{z^n}{\mu}\int_0^1 \phi(rz) r^{n+(1/\mu)-1} dr.$$

Note that if we let $\eta = rz$, then

$$1 + \frac{z^n}{\mu}\int_0^1 \phi(rz) r^{n+(1/\mu)-1} dr = 1 + \frac{1}{\mu z^{1/\mu}}\int_0^z \phi(\eta)\eta^{n+(1/\mu)-1} d\eta.$$

So,

$$(1-\nu)\frac{f(z)}{z} + \nu f'(z) = 1 + \frac{1}{\mu z^{1/\mu}}\int_0^z \phi(\eta)\eta^{n+(1/\mu)-1} d\eta.$$

Now, set $\psi(z) = (1-\nu)f(z)/z + \nu f'(z)$. Then

$$\psi(z) = 1 + \frac{1}{\mu z^{1/\mu}}\int_0^z \phi(\eta)\eta^{n+(1/\mu)-1} d\eta$$
$$z^{1/\mu}\psi(z) = z^{1/\mu} + \frac{1}{\mu}\int_0^z \phi(\eta)\eta^{n+(1/\mu)-1} d\eta.$$

Similarly, now differentiate both sides to get

$$z^{1/\mu}\psi'(z) + \frac{z^{(1/\mu)-1}\psi(z)}{\mu} = \frac{z^{(1/\mu)-1}}{\mu} + \frac{z^{n+(1/\mu)-1}\phi(z)}{\mu}$$

or

$$\mu z\psi'(z) + \psi(z) - 1 = z^n\phi(z). \tag{2.8}$$

Since $\mu z\psi'(z) + \psi(z) - 1 = (1-\alpha+2\gamma)f(z)/z + (\alpha-2\gamma)f'(z) + \gamma z f''(z) - 1$, equation (2.8) is becomes

$$(1-\alpha+2\gamma)\frac{f(z)}{z} + (\alpha-2\gamma)f'(z) + \gamma z f''(z) - 1 = z^n\phi(z). \tag{2.9}$$

Now, let $p(z) = (1-\alpha+2\gamma)f(z)/z + (\alpha-2\gamma)f'(z) + \gamma z f''(z) - 1$ and since $\phi(z) = \frac{1}{\delta}\int_0^1 g(tz)t^{n+(1/\delta)-1}dt$, equation (2.9) becomes

$$p(z) = \frac{z^n}{\delta}\int_0^1 g(tz)t^{n+(1/\delta)-1}dt.$$

Note that if we take $u = tz$, then

$$\frac{z^n}{\delta}\int_0^1 g(tz)t^{n+(1/\delta)-1}dt = \frac{1}{\delta z^{1/\delta}}\int_0^z g(u)u^{n+(1/\delta)-1}\,du.$$

It follows that

$$p(z) = \frac{1}{\delta z^{1/\delta}}\int_0^z g(u)u^{n+(1/\delta)-1}\,du.$$

Differentiating both sides, we obtain

$$p(z) + \delta z p'(z) = z^n g(z). \tag{2.10}$$

Substituting back $p(z) = (1-\alpha+2\gamma)f(z)/z + (\alpha-2\gamma)f'(z) + \gamma z f''(z) - 1$ into equation (2.10) yields

$$\begin{aligned}\delta\gamma z^2 f'''(z) + [\gamma+\delta(\alpha-\gamma)]zf''(z) + [(\delta(1-\alpha+2\gamma)+(\alpha-2\gamma)]f'(z)&\\ +(1-\alpha+2\gamma)(1-\delta)\frac{f(z)}{z} - 1 = z^n g(z).&\end{aligned}$$

Taking modulus both sides and using (2.5), we have

$$\begin{aligned}
&\Big|\delta\gamma z^2 f'''(z)+[\gamma+\delta(\alpha-\gamma)]zf''(z)+[(\delta(1-\alpha+2\gamma)+(\alpha-2\gamma))f'(z)+\\
&\qquad(1-\alpha+2\gamma)(1-\delta)\frac{f(z)}{z}-1\Big|\\
&\qquad\qquad=|z|^n|g(z)|\\
&\qquad\qquad<\frac{(1+n\delta)(1+n\mu)(1+n\nu)(\nu(1-\beta)-2)}{\nu(n+1-\beta)}.
\end{aligned}$$

It follows from Theorem 2.16 that the function f lies in $\mathcal{S}^*(\beta)$. $\square$

CHAPTER 3

SUBCLASSES OF CLOSE-TO-CONVEX FUNCTIONS

3.1 Introduction

Recall from Definition 1.6 that an analytic function f is said to be close-to-convex in the unit disk $\mathbb{D}$ if there exists a convex function g such that

$$\Re\left\{\frac{f'(z)}{g'(z)}\right\} > 0, \quad z \in \mathbb{D}.$$

In this chapter, a new subclass of close-to-convex functions is introduced and properties of this class of functions are discussed. This new class of functions is motivated from the work of Sakaguchi [47].

3.2 Motivation

In 1959, Sakaguchi [47] introduced and investigated the class of function starlike with respect to symmetric points in the unit disk $\mathbb{D}$. Let the class of these functions be denoted by $\mathcal{S}_s^*$.

Definition 3.1. [47] A function $f \in \mathcal{A}$ is said to be *starlike with respect to symmetric points* in $\mathbb{D}$ if for every $0 \le r < 1$ sufficiently close to 1, and every z_0 on the circle $|z| = r$, the angular velocity of $f(z)$ about $f(-z_0)$ is positive at $z = z_0$ as z traverses the circle $|z| = r$ in the positive direction.

Geometrically, the angular velocity of $f(z)$ about $f(-z_0)$ is given by

$$\begin{aligned}
\frac{d}{dt}\arg(f(z)-f(-z_0)) &= \frac{d}{dt}\operatorname{Im}[\ln(f(z)-f(-z_0))], \quad t\in[a,b] \\
&= \operatorname{Im}\left[\frac{d}{dt}\ln(f(z)-f(-z_0))\right] \\
&= \operatorname{Im}\left[\frac{d}{dz}\ln(f(z)-f(-z_0))\frac{dz}{dt}\right] \\
&= \operatorname{Im}\left[\frac{f'(z)}{f(z)-f(-z_0)}\frac{dz}{dt}\right].
\end{aligned}$$

On the circle $z=re^{it}, 0\le t\le 2\pi$, we have $z'(t)=ire^{it}=iz$. Hence,

$$\operatorname{Im}\left[\frac{f'(z)}{f(z)-f(-z_0)}\frac{dz}{dt}\right] = \operatorname{Im}\left[i\frac{zf'(z)}{f(z)-f(-z_0)}\right] = \Re\left[\frac{zf'(z)}{f(z)-f(-z_0)}\right].$$

By Definition 3.1, at the point $z=z_0$, the angular velocity of $f(z)$ about $f(-z_0)$ is positive, that is,

$$\Re\left[\frac{z_0f'(z_0)}{f(z_0)-f(-z_0)}\right]>0.$$

Since z_0 is arbitrary, we have

$$\Re\left[\frac{zf'(z)}{f(z)-f(-z)}\right]>0.$$

Theorem 3.1. [47] Let $f\in\mathcal{A}$. Then $f\in\mathcal{S}_s^*$ if and only if

$$\Re\left\{\frac{zf'(z)}{f(z)-f(-z)}\right\}>0, \quad z\in\mathbb{D}. \tag{3.1}$$

One of the example of $f\in\mathcal{S}_s^*$ is the odd starlike functions. Suppose $f\in\mathcal{A}$ is an odd starlike function. Then $f(-z)=-f(z)$ and $\Re\{zf'(z)/f(z)\}>0$ for all $z\in\mathbb{D}$.

Also,

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} = 2\Re\left\{\frac{zf'(z)}{2f(z)}\right\} = 2\Re\left\{\frac{zf'(z)}{f(z)+f(z)}\right\} = 2\Re\left\{\frac{zf'(z)}{f(z)-f(-z)}\right\} > 0.$$

Hence, the class of odd functions starlike with respect to origin is also in the class $\mathcal{S}_s^*$.

It is interesting to see that $f \in \mathcal{S}_s^*$ is also close-to-convex in $\mathbb{D}$ because the function $(f(z)-f(-z))/2$ is starlike in $\mathbb{D}$. The constant $1/2$ is for normalization purpose.

Theorem 3.2. [8] If $f \in \mathcal{S}_s^*$, then

$$F(z) = \frac{f(z)-f(-z)}{2} \in \mathcal{S}^*, \quad z \in \mathbb{D}.$$

Proof. We need to prove $\Re\{zF'(z)/F(z)\} > 0$ for $z \in \mathbb{D}$. Note that

$$F'(z) = \frac{f'(z)+f'(-z)}{2},$$

and so

$$\begin{aligned}\Re\left\{\frac{zF'(z)}{F(z)}\right\} &= \Re\left\{\frac{z(f'(z)+f'(-z))}{2}\frac{2}{f(z)-f(-z)}\right\}\\ &= \Re\left\{\frac{zf'(z)}{f(z)-f(-z)}\right\} + \Re\left\{\frac{zf'(-z)}{f(z)-f(-z)}\right\}.\end{aligned}$$

Since $f \in \mathcal{S}_s^*$, by Theorem 3.1, we have

$$\Re\left\{\frac{zf'(z)}{f(z)-f(-z)}\right\} > 0.$$

On the other hand,

$$\Re\left\{\frac{zf'(-z)}{f(z)-f(-z)}\right\} = \Re\left\{\frac{-(-z)f'(-z)}{f(z)-f(-z)}\right\} = \Re\left\{\frac{(-z)f'(-z)}{f(-z)-f(z)}\right\}.$$

Let $u=-z$. Since $z\in\mathbb{D}$, it follows that $u=-z$ is also in $\mathbb{D}$. Using Theorem 3.1, we obtain

$$\Re\left\{\frac{(-z)f'(-z)}{f(-z)-f(z)}\right\} = \Re\left\{\frac{uf'(u)}{f(u)-f(-u)}\right\} > 0.$$

Since u is an arbitrary point in $\mathbb{D}$, it follows that

$$\Re\left\{\frac{zf'(-z)}{f(z)-f(-z)}\right\} > 0.$$

Hence, $\Re\{zF'(z)/F(z)\} > 0$. $\square$

Motivated by Sakaguchi's class of starlike functions with respect to symmetric points, Gao and Zhou [13] discussed the class $\mathcal{K}_s$ of close-to-convex functions.

Definition 3.2. [13] A function $f\in\mathcal{A}$ belongs to $\mathcal{K}_s$ if there exists a function $g\in\mathcal{S}^*(1/2)$ such that

$$\Re\left\{\frac{-z^2f'(z)}{g(z)g(-z)}\right\} > 0, \quad z\in\mathbb{D}. \tag{3.2}$$

The idea used by Gao and Zhou is to replace the function $f(z)-f(-z)$ in the denominator of (3.1) by the $-g(z)g(-z)$, and the factor z is included for normalization, so that $-z^2f'(z)/(g(z)g(-z))$ takes the value 1 at $z=0$. To ensure the univalency of f, it is further assumed that g is starlike of order $1/2$ so that the function $-g(z)g(-z)/z$ is starlike (to be shown below), which implies the close-to-convexity of f.

Theorem 3.3. [13] Let $g \in \mathcal{S}^*(1/2)$. Then $-g(z)g(-z)/z \in \mathcal{S}^*$. Moreover, the function $-g(z)g(-z)/z$ is an odd starlike function.

Proof. Set $\phi(z) = -g(z)$ and $\psi(z) = g(-z)$. Since $g \in \mathcal{S}^*(1/2)$, it follows that $\Re\{zg'(z)/g(z)\} > 1/2$. Thus, we have

$$\Re\left\{\frac{z\phi'(z)}{\phi(z)}\right\} = \Re\left\{\frac{-zg'(z)}{-g(z)}\right\} = \Re\left\{\frac{zg'(z)}{g(z)}\right\} > \frac{1}{2}.$$

Also

$$\Re\left\{\frac{z\psi'(z)}{\psi(z)}\right\} = \Re\left\{\frac{(-z)g'(-z)}{g(-z)}\right\} > \frac{1}{2}$$

because $z \in \mathbb{D}$ implies $-z$ is also in $\mathbb{D}$. Let $F(z) = \phi(z)\psi(z)/z = -g(z)g(-z)/z$. Then, we have

$$\frac{zF'(z)}{F(z)} = \frac{z\phi'(z)}{\phi(z)} + \frac{z\psi'(z)}{\psi(z)} - 1$$

and so

$$\Re\left\{\frac{zF'(z)}{F(z)}\right\} = \Re\left\{\frac{z\phi'(z)}{\phi(z)}\right\} + \Re\left\{\frac{z\psi'(z)}{\psi(z)}\right\} - 1 > \frac{1}{2} + \frac{1}{2} - 1 = 0.$$

Since

$$F(-z) = \frac{-g(-z)g(z)}{-z} = -\left(\frac{-g(z)g(-z)}{z}\right) = -F(z)$$

for all $z \in \mathbb{D}$, it follows that $-g(z)g(-z)/z$ is an odd function. $\square$

For this class $\mathcal{K}_s$, Gao and Zhou [13] obtained several results such as inclusion relationships, sharp coefficient bounds, distortion theorem and radius of convexity. Later, Kowalczyk and Leś-Bomba [23] extended Definition 3.2 as follows.

Definition 3.3. [23] A function $f \in \mathcal{A}$ belongs to $\mathcal{K}_s(\alpha), 0 \leq \alpha < 1$, if there exists a function $g \in \mathcal{S}^*(1/2)$ such that

$$\Re\left\{\frac{-z^2 f'(z)}{g(z)g(-z)}\right\} > \alpha, \quad z \in \mathbb{D}.$$

They also showed that the class $\mathcal{K}_s(\alpha), 0 \leq \alpha < 1$ is associated with an appropriate subordination.

Theorem 3.4. [23] A function $f \in \mathcal{A}$ belongs to the class $\mathcal{K}_s(\alpha), 0 \leq \alpha < 1$, if and only if there exists a function $g \in \mathcal{S}^*(1/2)$ such that

$$\frac{-z^2 f'(z)}{g(z)g(-z)} \prec \frac{1+(1-2\alpha)z}{1-z}, \quad z \in \mathbb{D}.$$

Motivated by Kowalczyk and Leś-Bomba [23] works, Şeker [49] introduced a new class $\mathcal{K}_s^{(k)}(\alpha)$, where $0 \leq \alpha < 1$ and $k \geq 1$.

Definition 3.4. [49] A function $f \in \mathcal{A}$ belongs to $\mathcal{K}_s^{(k)}(\alpha)$, if there exists a function $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{S}^*((k-1)/k)$ such that

$$\Re\left\{\frac{z^k f'(z)}{g_k(z)}\right\} > \alpha, \quad z \in \mathbb{D},$$

where $0 \leq \alpha < 1$, $k \geq 1$ is a fixed positive integer and $g_k(z)$ is given by

$$g_k(z) = \prod_{v=0}^{k-1} \varepsilon^{-v} g(\varepsilon^v z), \tag{3.3}$$

with $\varepsilon = e^{2\pi i/k}$.

Let $G_k(z) = g_k(z)/z^{k-1}$. Then, we have

$$\begin{aligned} G_k(z) = \frac{g_k(z)}{z^{k-1}} &= \frac{\prod_{v=0}^{k-1} \varepsilon^{-v} g(\varepsilon^v z)}{z^{k-1}} \\ &= \frac{\prod_{v=0}^{k-1} \varepsilon^{-v} [\varepsilon^v z + \sum_{n=2}^{\infty} b_n (\varepsilon^v z)^n]}{z^{k-1}} \\ &= \frac{\prod_{v=0}^{k-1} [z + \sum_{n=2}^{\infty} b_n \varepsilon^{(n-1)v} z^n]}{z^{k-1}} \\ &= z + \sum_{n=2}^{\infty} B_n z^n. \end{aligned}$$

The function G_k is normalized in $\mathbb{D}$ because $G_k(0) = 0$ and $G_k'(0) = 1$.

For $k = 1$, we have

$$g_1(z) = \prod_{v=0}^{0} \varepsilon^0 g(\varepsilon^0 z) = g(z).$$

For $k = 2$, we have $\varepsilon = e^{2\pi i/2} = e^{\pi i} = \cos\pi + i \sin\pi = -1$ and

$$g_2(z) = \prod_{v=0}^{1} \varepsilon^{-v} g(\varepsilon^v z) = [g_1(z)][\varepsilon^{-1} g(\varepsilon z)] = -g_1(z) g(-z) = -g(z) g(-z).$$

Therefore, we have $\mathcal{K}_s^{(2)}(\alpha) = \mathcal{K}_s(\alpha)$, the class studied by Kowalczyk and Leś-Bomba [23]. Furthermore, for $\alpha = 0$, we have $\mathcal{K}_s^{(2)}(0) = \mathcal{K}_s$, the class defined by Gao in the paper [13].

Şeker [49] also showed that the class $\mathcal{K}_s^{(k)}(\alpha)$ is associated with an appropriate subordination.

Theorem 3.5. [49] A function $f(z) \in \mathcal{K}_s^{(k)}(\alpha)$ if and only if there exists $g \in \mathcal{S}^*(\frac{k-1}{k})$ such that

$$\frac{z^k f'(z)}{g_k(z)} \prec \frac{1 + (1 - 2\alpha) z}{1 - z}, \quad z \in \mathbb{D}.$$

Instead of requiring the quantity $-z^2f'(z)/(g(z)g(-z))$ subordinate to a particular positive real part function, Wang *et al.*[54] introduced a general class $\mathcal{K}_s(\varphi)$, where φ is any positive real part function in $\mathbb{D}$.

Definition 3.5. [54] For a normalized function φ with positive real part, the class $\mathcal{K}_s(\varphi)$ consists of function $f \in \mathcal{A}$ satisfying

$$\frac{-z^2f'(z)}{g(z)g(-z)} \prec \varphi(z)$$

for some function $g \in \mathcal{S}^*(1/2)$.

They proved the sharp distortion and growth distortion theorems and sufficient conditions in terms of the coefficient in the class $\mathcal{K}_s(\varphi)$. Later, Cho *et al.*[7] obtained a sharp estimate for the Fekete-Szegö functional, corresponding problem for the inverse function and distortion and growth theorem for functions belonging to the class $\mathcal{K}_s(\varphi)$.

Using a second order differential inequality, Wang and Chen [53] defined another new subclass of close-to-convex functions.

Definition 3.6. [53] A function $f \in \mathcal{A}$ is in the class $\mathcal{K}_s(\lambda, A, B)$ if it satisfies

$$\frac{-z^2f'(z)+\lambda z^3 f''(z)}{g(z)g(-z)} \prec \frac{1+Az}{1+Bz} \quad z \in \mathbb{D},$$

where $0 \leq \lambda \leq 1, -1 \leq B < A \leq 1$ and $g \in \mathcal{S}^*(1/2)$.

For the class $\mathcal{K}_s(\lambda, A, B)$, several results such as inclusion relationships, coefficient estimates, covering theorem and distortion theorem are derived.

Recently, using third order differential inequality, Goyal and Singh [16] introduced and studied the following subclass of analytic functions.

Definition 3.7. [16] For a normalized analytic function φ with positive real part, a function $f \in \mathcal{A}$ is in the class $\mathcal{K}_s(\lambda,\mu,\varphi)$ if it satisfies

$$\frac{z^2 f'(z) + (\lambda - \mu + 2\lambda\mu) z^3 f''(z) + \lambda\mu z^4 f'''(z)}{-g(z)g(-z)} \prec \varphi(z), \quad z \in \mathbb{D}$$

where $0 \leq \mu \leq \lambda \leq 1$ and $g \in \mathcal{S}^*(1/2)$.

For the class $\mathcal{K}_s(\lambda,\mu,\varphi)$, the results of coefficient estimates and Fekete-Szegö inequality are obtained. Obviously for $\mu = 0$ and $\varphi(z) = (1+Az)/(1+Bz)$ where $-1 \leq B < A \leq 1$, we get the class $\mathcal{K}_s(\lambda,A,B) := \mathcal{K}_s(\lambda,0,(1+Az)/(1+Bz))$ which was studied by Wang and Chen [53]. For $\lambda = \mu = 0$, we get the class $\mathcal{K}_s(\varphi) := \mathcal{K}_s(0,0,\varphi)$ which was studied by Cho *et al.*[7].

3.3 Main results

Motivated by aforementioned works, we now introduce the following subclass of analytic functions.

Definition 3.8. Let φ be an analytic normalized function with positive real part, $g \in \mathcal{S}^*((k-1)/k)$, and $g_k(z) = \prod_{v=0}^{k-1} \varepsilon^{-v} g(\varepsilon^v z)$ and $0 \leq \mu \leq \lambda \leq 1$. A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{K}_s^{(k)}(\lambda,\mu,\varphi)$ if

$$\frac{z^k f'(z) + (\lambda - \mu + 2\lambda\mu) z^{k+1} f''(z) + \lambda\mu z^{k+2} f'''(z)}{g_k(z)} \prec \varphi(z), \quad z \in \mathbb{D},$$

holds for some postive integer k.

For $k=2$, we have the class $\mathcal{K}_s^{(2)}(\lambda,\mu,\varphi) := \mathcal{K}_s(\lambda,\mu,\varphi)$ which was discussed by Goyal *et al.*[16].

The following lemmas are needed to prove the results later.

Lemma 3.1. [54] If $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{S}^*((k-1)/k)$, then

$$G_k(z) = \frac{g_k(z)}{z^{k-1}} = z + \sum_{n=2}^{\infty} B_n z^n \in \mathcal{S}^*. \tag{3.4}$$

Lemma 3.2. [46] Let $f(z) = 1 + \sum_{k=1}^{\infty} c_k z^k$ be analytic in $\mathbb{D}$ and $g(z) = 1 + \sum_{k=1}^{\infty} d_k z^k$ be analytic and convex in $\mathbb{D}$. If $f \prec g$, then

$$|c_k| \leq |d_1| \quad \text{where} \quad k \in \mathbb{N} := \{1,2,3,\ldots\}.$$

We first prove the inclusion relationships for the class $\mathcal{K}_s^{(k)}(\lambda,\mu,\varphi)$.

Theorem 3.6. Let $0 \leq \mu \leq \lambda \leq 1$ and φ be an analytic normalized function with positive real part. Then

$$\mathcal{K}_s^{(k)}(\lambda,\mu,\varphi) \subset \mathcal{K}.$$

Proof. Consider $f \in \mathcal{K}_s^{(k)}(\lambda,\mu,\varphi)$. By Definition 3.8, we have

$$\frac{z^k f'(z) + (\lambda - \mu + 2\lambda\mu) z^{k+1} f''(z) + \lambda\mu z^{k+2} f'''(z)}{g_k(z)} \prec \varphi(z),$$

which can be written as

$$\frac{zF'(z)}{G_k(z)} \prec \varphi(z)$$

where

$$F'(z) = f'(z) + (\lambda - \mu + 2\lambda\mu)zf''(z) + \lambda\mu z^2 f'''(z) \tag{3.5}$$

and $G_k(z)$ is defined in (3.4). Integrating $F'(z)$ in (3.5) leads to

$$\begin{aligned}
F(z) &= \int_0^z F'(t)\,dt \\
&= \int_0^z \left[f'(t) + (\lambda - \mu + 2\lambda\mu)tf''(t) + \lambda\mu z^2 f'''(t)\right]\,dt \\
&= \int_0^z f'(t)\,dt + (\lambda - \mu + 2\lambda\mu)\int_0^z tf''(t)\,dt + \int_0^z \lambda\mu t^2 f'''(t)\,dt \\
&= f(z) - f(0) + (\lambda - \mu + 2\lambda\mu)\int_0^z tf''(t)\,dt + \lambda\mu\left[\left[t^2 f''(t)\right]\Big|_0^z - 2\int_0^z tf''(t)\,dt\right] \\
&= f(z) + (\lambda - \mu)\int_0^z tf''(t)\,dt + \lambda\mu z^2 f''(z) \\
&= f(z) + (\lambda - \mu)\left[tf'(t) - f(t)\right]\Bigg|_0^z + \lambda\mu z^2 f''(z) \\
&= f(z) + (\lambda - \mu)(zf'(z) - f(z)) + \lambda\mu z^2 f''(z) \\
&= (1 - \lambda + \mu)f(z) + (\lambda - \mu)zf'(z) + \lambda\mu z^2 f''(z).
\end{aligned}$$

Thus

$$F(z) = (1 - \lambda + \mu)f(z) + (\lambda - \mu)zf'(z) + \lambda\mu z^2 f''(z).$$

Since $\Re\{\varphi(z)\} > 0$, we have

$$\Re\left\{\frac{zF'(z)}{G_k(z)}\right\} > 0.$$

Also, since $G_k(z) \in \mathcal{S}^*$ (by Lemma 3.1), by Definition 1.6, we deduce that

$$F(z) = (1 - \lambda + \mu)f(z) + (\lambda - \mu)zf'(z) + \lambda\mu z^2 f''(z) \in \mathcal{K}.$$

In order to show $f \in \mathcal{K}$, we consider three cases:

Case 1: $\mu = \lambda = 0$. It is then obvious that $f = F \in \mathcal{K}$.

Case 2: $\mu = 0, \lambda \neq 0$. Then we obtain

$$F(z) = (1-\lambda)f(z) + \lambda z f'(z). \tag{3.6}$$

Divide all the terms by λz in (3.6), we obtain

$$f'(z) + \frac{(1-\lambda)f(z)}{\lambda z} = \frac{F(z)}{\lambda z}. \tag{3.7}$$

To solve equation (3.7), an integrating factor

$$e^{\int \frac{(1-\lambda)dz}{\lambda z}} = e^{\frac{1-\lambda}{\lambda}\int \frac{dz}{z}} = \left(e^{\ln z}\right)^{\frac{1-\lambda}{\lambda}} = z^{(1/\lambda)-1}$$

is needed. Multiply $z^{(1/\lambda)-1}$ in (3.7), we have

$$z^{(1/\lambda)-1}\left(f'(z) + \frac{(1-\lambda)f(z)}{\lambda z}\right) = \frac{z^{(1/\lambda)-1}F(z)}{\lambda z}$$

$$\left(z^{(1/\lambda)-1}f(z)\right)' = \frac{z^{(1/\lambda)-2}F(z)}{\lambda}.$$

Solving for $f(z)$, we obtain

$$f(z) = \frac{z^{1-1/\lambda}}{\lambda}\int_0^z t^{(1/\lambda)-2}F(t)\,dt.$$

Taking $\gamma = (1/\lambda) - 1$, we have

$$f(z) = \frac{1+\gamma}{z^{\gamma}} \int_0^z F(t) t^{\gamma-1}\, dt.$$

Since $\Re\{\gamma\} = \Re\{(1/\lambda) - 1\} \geq 0$, by using Theorem 2.3, we conclude that $f \in \mathcal{K}$.

Case 3: $0 < \mu \leq \lambda \leq 1$. Then we have

$$F(z) = (1-\lambda+\mu) f(z) + (\lambda-\mu) z f'(z) + \lambda\mu z^2 f''(z) \in \mathcal{K}.$$

Let $G(z) = \frac{1}{(1-\lambda+\mu)} F(z)$, so $G(z) \in \mathcal{K}$. Then

$$G(z) = f(z) + \alpha z f'(z) + \beta z^2 f''(z) \tag{3.8}$$

where $\alpha = \frac{\lambda-\mu}{1-\lambda+\mu}$ and $\beta = \frac{\lambda\mu}{1-\lambda+\mu}$. Consider $\delta > 0$ and $\nu > 0$ satisfying

$$\delta + \nu = \alpha - \beta \quad \text{and} \quad \delta\nu = \beta.$$

The equation (3.8) can be written as

$$G(z) = f(z) + (\delta + \nu + \delta\nu) z f'(z) + \delta\nu z^2 f''(z).$$

Let $p(z) = f(z) + \delta z f'(z)$, then

$$p(z) + \nu z p'(z) = f(z) + (\delta + \nu + \delta\nu) z f'(z) + \delta\nu z^2 f''(z) = G(z).$$

On the other hand, $p(z)+\nu zp'(z)=\nu z^{1-1/\nu}\left(z^{1/\nu}p(z)\right)'$. So,

$$G(z)=\nu z^{1-1/\nu}\left[\delta z^{1+(1/\nu)-1/\delta}\left(z^{1/\delta}f(z)\right)'\right]'.$$

Hence

$$\delta z^{1+(1/\nu)-1/\delta}\left(z^{1/\delta}f(z)\right)'=\frac{1}{\nu}\int_0^z w^{(1/\nu)-1}G(w)\,dw.$$

Multiply by $(1+\nu)$ both sides and divide by $z^{1/\nu}$ to get

$$(1+\nu)\delta z^{1-1/\delta}\left(z^{1/\delta}f(z)\right)'=\frac{1+1/\nu}{z^{1/\nu}}\int_0^z w^{(1/\nu)-1}G(w)\,dw:=H(z).$$

Taking $\gamma=1/\nu$, then $\Re\{\gamma\}\geq 0$, and by using Theorem 2.3, we conclude that $H\in\mathcal{K}$.

Further,

$$(1+\nu)z^{1/\delta}f(z)=\frac{1}{\delta}\int_0^z t^{(1/\delta)-1}H(t)dt.$$

Multiply by $(1+\delta)$ both sides and divide by $z^{1/\delta}$ to obtain

$$(1+\delta)(1+\nu)f(z)=\frac{1+1/\delta}{z^{1/\delta}}\int_0^z t^{(1/\delta)-1}H(t)dt.$$

Taking $\gamma=1/\delta$, then $\Re\{\gamma\}\geq 0$, and by using Theorem 2.3, we conclude that $f\in\mathcal{K}$. $\square$

Next, we give the coefficient estimates of functions in the class $\mathcal{K}_s^{(k)}(\lambda,\mu,\varphi)$.

Theorem 3.7. Let $0\leq\mu\leq\lambda\leq 1$ and $\varphi(z)$ is a normalized analytic convex function in $\mathbb{D}$. If $f \in \mathcal{K}_s^{(k)}(\lambda, \mu, \varphi)$, then

$$|a_n| \leq \frac{1}{1+(n-1)(\lambda-\mu+n\lambda\mu)}\left(1+\frac{|\varphi'(0)|(n-1)}{2}\right), \quad n \in \mathbb{N}.$$

Proof. From the definition of $\mathcal{K}_s^{(k)}(\lambda, \mu, \varphi)$, we know that there exists a function with positive real part p such that

$$p(z) = \frac{zF'(z)}{G_k(z)}$$

where $F'(z) = f'(z) + (\lambda - \mu + 2\lambda\mu)zf''(z) + \lambda\mu z^2 f'''(z)$ and $G_k(z) = z + \sum_{n=2}^{\infty} B_n z^n$. Hence, we have

$$zf'(z) + z^2 f''(z)(\lambda - \mu + 2\lambda\mu) + \lambda\mu z^3 f'''(z) = p(z)G_k(z). \tag{3.9}$$

Expanding both sides of (3.9), we obtain

$$\begin{aligned} z + \sum_{n=2}^{\infty} na_n z^n + (\lambda - \mu + 2\lambda\mu)\sum_{n=2}^{\infty} n(n-1)a_n z^n + \lambda\mu \sum_{n=2}^{\infty} n(n-1)(n-2)a_n z^n \\ = z + \sum_{n=2}^{\infty} B_n z^n + \sum_{n=1}^{\infty} p_n z^{n+1} + \left(\sum_{n=1}^{\infty} p_n z^n\right)\left(\sum_{n=2}^{\infty} B_n z^n\right). \end{aligned}$$

Comparing the coefficient of z^n, we get

$$na_n\left[1+(n-1)(\lambda-\mu+n\lambda\mu)\right] = B_n + p_{n-1} + p_1 B_{n-1} + \cdots + p_{n-2}B_2. \tag{3.10}$$

Since $G_k(z)$ is starlike, we have

$$|B_n| \leq n. \tag{3.11}$$

Also, by Lemma 3.2, we know that

$$|p_n| = \left|\frac{p^{(n)}(0)}{n!}\right| \leq |\varphi'(0)| \quad (n \in \mathbb{N}). \tag{3.12}$$

Combining (3.10), (3.11) and (3.12), we obtain

$$n|a_n|\big[1+(n-1)(\lambda-\mu+n\lambda\mu)\big] \leq n+|\varphi'(0)|+|\varphi'(0)|\sum_{k=2}^{n-1} k.$$

Note that

$$\sum_{k=2}^{n-1} k = \frac{n-2}{2}\left[2+(n-1)\right] = \frac{(n-2)(n+1)}{2}.$$

Hence, we obtain

$$|a_n| \leq \frac{1}{1+(n-1)(\lambda-\mu+n\lambda\mu)}\left(1+\frac{|\varphi'(0)|(n-1)}{2}\right). \qquad \square$$

For $\mu = 0$ in Definition (3.8), we have the class $\mathcal{K}(\lambda, 0, \varphi := \mathcal{K}_s^{(k)}(\lambda, \varphi)$ which is defined as

$$\mathcal{K}_s^{(k)}(\lambda,\varphi) = \left\{ f \in \mathcal{A} : \frac{z^k f'(z) + \lambda z^{k+1} f''(z)}{g_k(z)} \prec \varphi(z), \quad z \in \mathbb{D} \right\}.$$

Therefore, the corresponding coefficient estimate is by setting $\mu = 0$ in Theorem (3.7) yield

Corollary 3.1. If $f = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{K}_s^{(k)}(\lambda, \varphi)$, then

$$|a_n| \leq \frac{1}{1+\lambda(n-1)}\left(1+\frac{|\varphi'(0)|(n-1)}{2}\right), \quad n \in \mathbb{N}.$$

Furthermore, by setting $\lambda = 0 = \mu$ in Definition (3.8), we have the class $\mathcal{K}_s^{(k)}$ which is defined as

$$\mathcal{K}_s^{(k)} = \left\{ f \in \mathcal{A} : \frac{z^k f'(z)}{g_k(z)} \prec \varphi(z), \quad z \in \mathbb{D} \right\}.$$

Set $\lambda = 0$ in Corollary 3.1 will yield

Corollary 3.2. If $f = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{K}_s^{(k)}(\varphi)$, then

$$|a_n| \le \left(1 + \frac{|\varphi'(0)|(n-1)}{2}\right), \quad n \in \mathbb{N}.$$

The Fekete-Szegö coefficient functional for normalized analytic univalent functions is well known for its rich history in the theory of geometric function theory. This functional also arises naturally in the investigation of the univalency of analytic functions. A classical theorem of Fekete and Szegö [11] states that, for $f \in \mathcal{S}$,

$$|a_3 - \mu a_2^2| = \begin{cases} 3 - 4\mu & \text{if } \mu \le 0, \\ 1 + 2e^{-2\mu/(1-\mu)} & \text{if } 0 \le \mu < 1 \cdot \\ 4\mu - 3 & \text{if } \mu > 1 \end{cases} \tag{3.13}$$

This inequality is sharp for each μ. Later, Pfluger [43] considered the problem when μ is a complex number. He showed that the inequality (3.13) holds for complex μ such that $\Re\, \mu/(1-\mu) \ge 0$. Keogh and Merkes [20], Koepf [22] and London [28] obtained the solution of the Fekete-Szegö problem for the class of close-to-convex functions.

In this section, we obtain the Fekete-Szegö inequality for functions in $\mathcal{K}_s^{(k)}(\lambda, \mu, \varphi)$. To prove our result, the following lemmas are needed.

Lemma 3.3. [20] If $p(z) = 1 + c_1 z + c_2 z^2 + \dots$ is a function with positive real part, then for any complex number μ,

$$|c_2 - \mu c_1^2| \leq 2\max\{1, |2\mu - 1|\}.$$

This inequality is sharp for the Mobius function, $m(z) = (1+z)/(1-z)$ if $|2\mu - 1| \geq 1$ and for $m(z^2) = (1+z^2)/(1-z^2)$ if $|2\mu - 1| \leq 1$.

Lemma 3.4. [22] Let $g(z) = z + b_2 z^2 + b_3 z^3 + \dots \in \mathcal{S}^*$. Then, for any $\lambda \in \mathbb{C}$,

$$|b_3 - \lambda b_2^2| \leq \max\{1, |3 - 4\lambda|\}.$$

This inequality is sharp for the Koebe function, $k(z) = z/(1-z)^2$ if $|3 - 4\lambda| \geq 1$ and for $(k(z)^2)^{1/2} = z/(1-z^2)$ if $|3 - 4\lambda| \leq 1$.

Theorem 3.8. Let $\varphi(z) = 1 + q_1 z + q_2 z^2 + \dots$ be a normalized analytic function with positive real part on $\mathbb{D}$. For a function $f(z) = z + a_2 z^2 + a_3 z^3 + ...$ belonging to the class $\mathcal{K}_s^{(k)}(\lambda, \mu, \varphi)$ and $\delta \in \mathbb{C}$, the following estimate holds

$$\begin{aligned} |a_3 - \delta a_2^2| \leq \frac{\max\{1, |3 - 4\alpha|\} + q_1 \max\{1, |2\beta - 1|\}}{3(1 + 2\lambda - 2\mu + 6\lambda\mu)} \\ + 2q_1 \left(\frac{1}{3(1 + 2\lambda - 2\mu + 6\lambda\mu)} - \frac{\mu}{2(1 + \lambda - \mu + 2\lambda\mu)^2} \right) \end{aligned}$$

where

$$\alpha = \frac{3\delta(1 + 2\lambda - 2\mu + 6\lambda\mu)}{4(1 + \lambda - \mu + 2\lambda\mu)}$$

and

$$\beta = \frac{1}{2}\left(1 - \frac{q_2}{q_1} - \frac{3\delta q_2^2 d_1^2 (1 + 2\lambda - 2\mu + 6\lambda\mu)}{4(1 + \lambda - \mu + 2\lambda\mu)^2)} \right).$$

Proof. Since $f \in \mathcal{K}_s^{(k)}(\lambda, \mu, \varphi)$, then there exists an analytic Schwarz function ω with $\omega(0) = 0$ and $|\omega(z)| < 1$ in $\mathbb{D}$ such that

$$\frac{z^k f'(z) + z^{k+1} f''(z)(\lambda - \mu + 2\lambda\mu) + \lambda\mu z^{k+2} f'''(z)}{g_k(z)} = \varphi(\omega(z)). \tag{3.14}$$

Define the function h by

$$h(z) = \frac{1+\omega(z)}{1-\omega(z)} = 1 + d_1 z + d_2 z^2 + \cdots.$$

Since ω is a Schwarz function, we see that $\Re\{h(z)\} > 0$ and $h(0) = 1$. Also, we have

$$\begin{aligned}
\varphi(\omega(z)) &= \varphi\left(\frac{h(z)-1}{h(z)+1}\right) \\
&= \varphi\left(\frac{d_1 z + d_2 z^2 + \dots}{2 + d_1 z + d_2 z^2 + \dots}\right) \\
&= \varphi\left(\frac{1}{2} d_1 z + \frac{1}{2}\left(d_2 - \frac{d_1^2}{2}\right) z^2 + \dots\right) \\
&= 1 + \frac{1}{2} q_1 d_1 z + \frac{1}{2} q_1 \left(d_2 - \frac{d_1^2}{2}\right) z^2 + \frac{1}{4} q_2 d_1^2 z^2 + \cdots
\end{aligned} \tag{3.15}$$

The series expansion of

$$\frac{z^k f'(z) + z^{k+1} f''(z)(\lambda - \mu + 2\lambda\mu) + \lambda\mu z^{k+2} f'''(z)}{g_k(z)}$$

is given by

$$\begin{aligned}
&1 + (2a_2(1 + \lambda + 2\lambda\mu - \mu) - B_2)z + \\
&\quad (3a_3(1 + 2\lambda + 6\lambda\mu - 2\mu) - 2a_2(1 + \lambda + 2\lambda\mu - \mu)B_2 + B_2^2 - B_3)z^2 + \cdots.
\end{aligned} \tag{3.16}$$

By comparing (3.15) and (3.16), we have

$$a_2 = \frac{2B_2 + q_1 d_1}{4(1+\lambda-\mu+2\lambda\mu)} \quad \text{and} \quad a_3 = \frac{2B_2 q_1 d_1 + 2q_1\left(d_2 - \frac{d_1^2}{2}\right) + q_2 d_1^2 + 4B_3}{12(1+2\lambda-2\mu+6\lambda\mu)}.$$

Therefore, we have

$$\begin{aligned} a_3 - \delta a_2^2 = {} & \frac{1}{3(1+2\lambda-2\mu+6\lambda\mu)}(B_3 - \alpha B_2^2) + \frac{q_1}{6(1+2\lambda-2\mu+6\lambda\mu)}(d_2 - \beta d_1^2) \\ & + \frac{B_2 q_1 d_1}{2}\left(\frac{1}{3(1+2\lambda-2\mu+6\lambda\mu)} - \frac{\delta}{2(1+\lambda-\mu+2\lambda\mu)^2}\right) \end{aligned}$$

where

$$\alpha = \frac{3\delta(1+2\lambda-2\mu+6\lambda\mu)}{4(1+\lambda-\mu+2\lambda\mu)^2}$$

and

$$\beta = \frac{1}{2}\left(1 - \frac{q_2}{q_1} - \frac{3\delta q_2^2 d_1^2(1+2\lambda-2\mu+6\lambda\mu)}{4(1+\lambda-\mu+2\lambda\mu)^2}\right).$$

Taking modulus both sides, we have

$$\begin{aligned} |a_3 - \delta a_2^2| = {} & \Bigg| \frac{(B_3 - \alpha B_2^2)}{3(1+2\lambda-2\mu+6\lambda\mu)} + \frac{q_1(d_2 - \beta d_1^2)}{6(1+2\lambda-2\mu+6\lambda\mu)} \\ & + \frac{B_2 q_1 d_1}{2}\left(\frac{1}{3(1+2\lambda-2\mu+6\lambda\mu)} - \frac{\mu}{2(1+\lambda-\mu+2\lambda\mu)^2}\right)\Bigg| \\ \leq {} & \frac{|B_3 - \alpha B_2^2|}{3(1+2\lambda-2\mu+6\lambda\mu)} + \frac{q_1|d_2 - \beta d_1^2|}{6(1+2\lambda-2\mu+6\lambda\mu)} \\ & + \frac{q_1|B_2||d_1|}{2}\left(\frac{1}{3(1+2\lambda-2\mu+6\lambda\mu)} - \frac{\mu}{2(1+\lambda-\mu+2\lambda\mu)^2}\right). \end{aligned}$$

Using Lemma 3.3 and Lemma 3.4, we obtain

$$|a_3 - \mu a_2^2| \leq \frac{\max\{1, |3-4\alpha|\}}{3(1+2\lambda-2\mu+6\lambda\mu)} + 2\frac{q_1 \max\{1, |2\beta-1|}{6(1+2\lambda-2\mu+6\lambda\mu)} + \frac{q_1|B_2||d_1|}{2}\left(\frac{1}{3(1+2\lambda-2\mu+6\lambda\mu)} - \frac{\mu}{2(1+\lambda-\mu+2\lambda\mu)^2}\right).$$

Also, using $|B_2| \leq 2$ and $|d_1| \leq 2$, the result is proved. □

By setting $\varphi(z) = (1+Az)/(1+Bz)$ in Definition 3.8, we get the class $\mathcal{K}_s^{(k)}(\lambda, \mu, A, B)$ where $-1 \leq B < A \leq 1$.

Definition 3.9. A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{K}_s^{(k)}(\lambda, \mu, A, B)$ if it satisfies

$$\frac{z^k f'(z) + z^{k+1} f''(z)(\lambda - \mu + 2\lambda\mu) + \lambda\mu z^{k+2} f'''(z)}{g_k(z)} \prec \frac{1+Az}{1+Bz} \tag{3.17}$$

where $0 \leq \mu \leq \lambda \leq 1$, $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{S}^*(\frac{k-1}{k})$, $k \geq 1$ is a fixed positive integer and $g_k(z)$ is defined by (3.3) with $\varepsilon = e^{2\pi i/k}$.

Using the concept of subordination in Definition 1.5, we have

$$\frac{z^k f'(z) + z^{k+1} f''(z)(\lambda - \mu + 2\lambda\mu) + \lambda\mu z^{k+2} f'''(z)}{g_k(z)} = \frac{1+A\omega(z)}{1+B\omega(z)}.$$

where $\omega(z)$ is analytic in $\mathbb{D}$ and satisfy $\omega(0) = 0$ and $|\omega(z)| < 1$. Let

$$p(z) = \frac{z^k f'(z) + z^{k+1} f''(z)(\lambda - \mu + 2\lambda\mu) + \lambda\mu z^{k+2} f'''(z)}{g_k(z)}.$$

Then $p(z)-1 = w(z)(A-Bp(z))$. Taking absolute both sides, we obtain

$$|p(z)-1| < |A-Bp(z)|.$$

Therefore, it can seen that the condition in (3.17) is equivalent to

$$\begin{aligned}
&\left|\frac{z^k f'(z)+z^{k+1}f''(z)(\lambda-\mu+2\lambda\mu)+\lambda\mu z^{k+2}f'''(z)}{g_k(z)}-1\right| \\
&\qquad < \left|A-\frac{B(z^k f'(z)+z^{k+1}f''(z)(\lambda-\mu+2\lambda\mu)+\lambda\mu z^{k+2}f'''(z))}{g_k(z)}\right| \qquad (3.18)
\end{aligned}$$

We prove sufficient condition for functions to belong to the class $\mathcal{K}_s^{(k)}(\lambda,\mu,A,B)$.

Theorem 3.9. Let $0 \leq \mu \leq \lambda \leq 1$ and $-1 \leq B < A \leq 1$. If $f \in \mathcal{A}$ satisfies the inequality

$$(1+|B|)\sum_{n=2}^{\infty} n[1+(n-1)(\lambda-\mu+n\lambda\mu)]|a_n| + (1+|A|)\sum_{n=2}^{\infty}|B_n| \leq A-B$$

and for $n = 2,3,\ldots$ the coefficients of B_n given by (3.4), then $f \in \mathcal{K}_s^{(k)}(\lambda,\mu,A,B)$.

Proof. Recall that

$$F'(z) = f'(z)+(\lambda-\mu+2\lambda\mu)zf''(z)+\lambda\mu z^2 f'''(z)$$

and

$$\frac{g_k(z)}{z^{k-1}} = z+\sum_{n=2}^{\infty} B_n z^n.$$

Now, let M denoted by

$$
\begin{aligned}
M &= \left| zF'(z) - \frac{g_k(z)}{z^{k-1}} \right| - \left| -\frac{Ag_k(z)}{z^{k-1}} - BzF'(z) \right| \\
&= \left| zf'(z) + (\lambda - \mu + 2\lambda\mu) z^2 f''(z) + \lambda\mu z^3 f'''(z) - \left(z + \sum_{n=2}^{\infty} B_n z^n \right) \right| \\
&\quad - \left| A\left(z + \sum_{n=2}^{\infty} B_n z^n \right) - B[zf'(z) + z^2 f''(z)(\lambda - \mu + 2\lambda\mu) + \lambda\mu z^3 f'''(z)] \right| \\
&= \left| \sum_{n=2}^{\infty} n a_n z^n [1 + (n-1)(\lambda - \mu + n\lambda\mu)] - \sum_{n=2}^{\infty} B_n z^n \right| \\
&\quad - \left| (A-B)z + A\sum_{n=2}^{\infty} B_n z^n - B\sum_{n=2}^{\infty} n a_n z^n [1 + (n-1)(\lambda - \mu + n\lambda\mu)] \right|.
\end{aligned}
$$

Then, for $|z| = r < 1$, we have

$$
\begin{aligned}
M &\leq \sum_{n=2}^{\infty} n[1 + (n-1)(\lambda - \mu + n\lambda\mu)]|a_n| r^n + \sum_{n=2}^{\infty} |B_n| r^n \\
&\quad - \left[(A-B)r - |A| \sum_{n=2}^{\infty} |B_n| r^n - |B| \sum_{n=2}^{\infty} n[1 + (n-1)(\lambda - \mu + n\lambda\mu)]|a_n| r^n \right] \\
&< \left[-(A-B) + (1+|B|) \sum_{n=2}^{\infty} n[1 + (n-1)(\lambda - \mu + n\lambda\mu)]|a_n| + (1+|A|) \sum_{n=2}^{\infty} |B_n| \right] r \\
&\leq 0.
\end{aligned}
$$

From the above calculation, we obtain $M < 0$. Thus, we have

$$
\begin{aligned}
&\left| zf'(z) + z^2 f''(z)(\lambda - \mu + 2\lambda\mu) + \lambda\mu z^3 f'''(z) - \tfrac{g_k(z)}{z^{k-1}} \right| \\
&\qquad < \left| A\frac{g_k(z)}{z^{k-1}} - B[zf'(z) + z^2 f''(z)(\lambda - \mu + 2\lambda\mu) + \lambda\mu z^3 f'''(z)] \right|
\end{aligned}
$$

which is equivalent to (3.18). Therefore, $f \in \mathcal{K}_s^{(k)}(\lambda, \mu, A, B)$. $\square$

Setting $\mu = 0$ in Theorem 3.8, we get

Corollary 3.3. Let $f \in \mathcal{A}$ and $-1 \leq B < A \leq 1$. If

$$(1+|B|)\sum_{n=2}^{\infty} n[1+\lambda(n-1)]|a_n| + (1+|A|)\sum_{n=2}^{\infty} |B_n| \leq A-B,$$

where B_n given by (3.4), then $f(z) \in \mathcal{K}_s^{(k)}(\lambda, A, B)$.

Further setting $\lambda = 0$ in Corollary 3.3 , we obtain

Corollary 3.4. Let $f \in \mathcal{A}$ and $-1 \leq B < A \leq 1$. If

$$(1+|B|)\sum_{n=2}^{\infty} n|a_n| + (1+|A|)\sum_{n=2}^{\infty} |B_n| \leq A-B,$$

where B_n given by (3.4), then $f \in \mathcal{K}_s^{(k)}(A, B)$.

Remark 3.1. By taking $A = \beta, B = -\alpha\beta$ in Corollary 3.4, we get the result obtained in [55, Theorem 5]. In addition, by taking $A = 1-2\gamma, B = -1$, we get the result obtained in [49, Theorem 2].

CHAPTER 4

CONCLUSION

Sufficient conditions to ensure starlikeness of analytic functions is an important area of research in geometric function theory. This can be done by using the differential inequalities, integral operators and others.

By using certain third-order differential inequality, sufficient conditions for an analytic function f, defined on the unit disk $\mathbb{D}$, to be starlike of order β, where $0 \leq \beta < 1$ is obtained. A new starlike function of order β which can be expressed in terms of the triple integral is constructed by virtue of the third order differential inequality.

Furthermore, a new subclass of close-to-convex functions is studied and several properties such as inclusion relationships, coefficient estimates, Fekete-Szegö inequality and sufficient conditions are obtained.

REFERENCES

[1] R. Aghalary and S. Joshi. Starlikeness of double integral operators. *Sarajevo J. Math.*, 10(22)(1):47–54, 2014.

[2] J. W. Alexander. Functions which map the interior of the unit circle upon simple regions. *Ann. of Math. (2)*, 17(1):12–22, 1915.

[3] S. D. Bernardi. Convex and starlike univalent functions. *Trans. Amer. Math. Soc.*, 135:429–446, 1969.

[4] L. Bieberbach. Uber die koeffizienten derjenigen potenzreihen, welche eine schlichte abbildung des einheitskreises vermitteln. *Semesterberichte Preussische Akademie der Wissenschaften*, 38:940–955, 1916.

[5] C. Carathéodory. Über den Variabilitätsbereich der Koeffizienten von Potenzreihen, die gegebene Werte nicht annehmen. *Math. Ann.*, 64(1):95–115, 1907.

[6] R. Chandrashekar, R.M. Ali, K. G. Subramanian, and A. Swaminathan. Starlikeness of functions defined by third-order differential inequalities and integral operators. *Abstr. Appl. Anal.*, pages Art. ID 723097, 6, 2014.

[7] N. E. Cho, Oh Sang Kwon, and V. Ravichandran. Coefficient, distortion and growth inequalities for certain close-to-convex functions. *Journal of Inequalities and Applications*, 2011(1):1–7, 2011.

[8] R. N. Das and P. Singh. On subclasses of schlicht mapping. *Indian J. Pure Appl. Math.*, 8(8):864–872, 1977.

[9] L. de Branges. A proof of the Bieberbach conjecture. *Acta Math.*, 154(1-2):137–152, 1985.

[10] P.L. Duren. *Univalent functions*, volume 259. Grundlehren der Mathematischen Wissenschaften, 1983.

[11] M. Fekete and G. Szegö. Eine bemerkung Über ungerade schlichte funktionen. *Journal of the London Mathematical Society*, s1-8(2):85–89, 1933.

[12] R. Fournier and P. Mocanu. Differential inequalities and starlikeness. *Complex Var. Theory Appl.*, 48(4):283–292, 2003.

[13] C. Gao and S. Zhou. On a class of analytic functions related to the starlike functions. *Kyungpook Math. J.*, 45(1):123–130, 2005.

[14] P. R. Garabedian and M. A. Schiffer. A proof of the Bieberbach conjecture for the fourth coefficient. *J. Rational Mech. Anal.*, 4:427–465, 1955.

[15] A.W. Goodman. *Univalent functions. Vol. I.* Mariner Publishing Co., Inc., Tampa, FL, 1983.

[16] S.P. Goyal and O. Singh. Certain subclasses of close-to-convex functions. *Vietnam J. Math.*, 42(1):53–62, 2014.

[17] D. J. Hallenbeck and Stephan Ruscheweyh. Subordination by convex functions. *Proc. Amer. Math. Soc.*, 52:191–195, 1975.

[18] R. Herglotz. Über potenzreihen mit positivem. *Oversikt av Finska Vetenskaps-Soc. Forh*, 63:501–511, 1911.

[19] W. Kaplan. Close-to-convex schlicht functions. *Michigan Math. J.*, 1:169–185, 1952.

[20] F. R. Keogh and E. P. Merkes. A coefficient inequality for certain classes of analytic functions. *Proc. Amer. Math. Soc.*, 20:8–12, 1969.

[21] P. Koebe. Ueber die uniformisierung beliebiger analytischer kurven (zweite mitteilung). *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse*, pages 633–669, 1907.

[22] W. Koepf. On the Fekete-Szegő problem for close-to-convex functions. *Proc. Amer. Math. Soc.*, 101(1):89–95, 1987.

[23] J. Kowalczyk and E. Leś-Bomba. On a subclass of close-to-convex functions. *Appl. Math. Lett.*, 23(10):1147–1151, 2010.

[24] K. Kuroki and S. Owa. Double integral operators concerning starlike of order β. *Int. J. Differ. Equ.*, pages Art. ID 737129, 13, 2009.

[25] Z. Lewandowski, S.S. Miller, and E. Złotkiewicz. Generating functions for some classes of univalent functions. *Proc. Amer. Math. Soc.*, 56:111–117, 1976.

[26] R. J. Libera. Some classes of regular univalent functions. *Proc. Amer. Math. Soc.*, 16:755–758, 1965.

[27] K. Löewner. Untersuchungen uber die verzerrung bei konformen abbildungen des einheitsreises $|z| < 1$. die durch funktionen mit nicht verschwindender ableitung geliefert werden. *Ber. Verh. Sachs. Ges. Wiss. Leipzig.*, 69:89–106, 1917.

[28] R. R. London. Fekete-Szegő inequalities for close-to-convex functions. *Proc. Amer. Math. Soc.*, 117(4):947–950, 1993.

[29] K. Löwner. Untersuchungen über schlichte konforme Abbildungen des Einheit-

skreises. I. *Math. Ann.*, 89(1-2):103–121, 1923.

[30] A. Marx. Untersuchungen uber schlichte abbildungen. *Math. Ann.*, 107:40–67, 1932.

[31] S.S. Miller and P.T. Mocanu. Double integral starlike operators. *Integral Transforms Spec. Funct.*, 19(7-8):591–597, 2008.

[32] S.S. Miller, P.T. Mocanu, and M.O. Reade. Starlike integral operators. *Pacific J. Math.*, 79(1):157–168, 1978.

[33] P. T. Mocanu. Two simple sufficient conditions for starlikeness. *Mathematica (Cluj)*, 34(57)(2):175–181, 1992.

[34] P.T. Mocanu. Some simple criteria for starlikeness and convexity. *Libertas Math.*, 13:27–40, 1993.

[35] R. Nevalinna. Uber die konforme abbildung von sterngebieten. *Oversikt av Finska Vetenskaps-Soc. Forh*, 63(A)(6):1–21, 1921.

[36] K. Noshiro. On the theory of schlicht functions. *J. Fac. Sci., Hokkaido Univ*, 3 (2):129–155, 1934.

[37] M. Obradović. Simple sufficient conditions for univalence. *Mat. Vesnik*, 49(3-4): 241–244, 1997.

[38] S. Ozaki. On the theory of multivalent functions. II. *Sci. Rep. Tokyo Bunrika Daigaku. Sect. A.*, A40:167–188, 1935.

[39] M. Ozawa. An elementary proof of the Bieberbach conjecture for the sixth coefficient. *Kōdai Math. Sem. Rep.*, 21:129–132, 1969.

[40] N. N. Pascu. Alpha-close-to-convex functions. In *Romanian-Finnish Seminar on Complex Analysis (Proc.,Bucharest, 1976)*, volume 743 of *Lecture Notes in Math.*, pages 331–335. Springer, Berlin, 1979.

[41] R. Pederson and M. Schiffer. A proof of the Bieberbach conjecture for the fifth coefficient. *Arch. Rational Mech. Anal.*, 45:161–193, 1972.

[42] Roger N. Pederson. A proof of the Bieberbach conjecture for the sixth coefficient. *Arch. Rational Mech. Anal.*, 31:331–351, 1968/1969.

[43] A. Pfluger. The Fekete-Szegő inequality for complex parameters. *Complex Variables Theory Appl.*, 7(1-3):149–160, 1986.

[44] M.O. Reade. On close-to-convex univalent functions. *Michigan Math. J.*, 3: 59–62, 1955.

[45] M.I.S. Robertson. On the theory of univalent functions. *Ann. of Math.*, 37(2): 374–408, 1936.

[46] W. Rogosinski. On the coefficients of subordinate functions. *Proc. London Math. Soc.*, 48(2):48–82, 1943.

[47] K. Sakaguchi. On a certain univalent mapping. *Journal of the Mathematical Society of Japan*, 11:72–75, 1959.

[48] A. Schild. On a class of univalent, star shaped mappings. *Proc. Amer. Math. Soc.*, 9:751–757, 1958.

[49] B. Şeker. On certain new subclass of close-to-convex functions. *Appl. Math. Comput.*, 218(3):1041–1045, 2011.

[50] E. Strohhäcker. Beiträge zur Theorie der schlichten Funktionen. *Math. Z.*, 37(1): 356–380, 1933.

[51] S. Verma, S. Gupta, and S. Singh. Starlikeness of a double integral operator. *Tamsui Oxf. J. Inf. Math. Sci.*, 29(3):351–359, 2013.

[52] S. Verma, S. Gupta, and S. Singh. Starlikeness of an integral transform. *Mathematica*, 56(79)(1):93–98, 2014.

[53] Z. Wang and D. Chen. On a subclass of close-to-convex functions. *Hacet. J. Math. Stat.*, 38(2):95–101, 2009.

[54] Z. Wang, C. Gao, and S. Yuan. On certain subclass of close-to-convex functions. *Acta Math. Acad. Paedagog. Nyházi. (N.S.)*, 22(2):171–177, 2006.

[55] Z. Wang, C. Gao, and S. Yuan. On certain new subclass of close-to-convex functions. *Mat. Vesnik*, 58(3-4):119–124, 2006.

[56] S. Warschawski. On the higher derivatives at the boundary in conformal mapping. *Trans. Amer. Math. Soc.*, 38(2):310–340, 1935.

LIST OF PUBLICATIONS

[1] Yao Liang Chung, See Keong Lee, and Maisarah Haji Mohd. On a subclass of close-to-convex functions. arXiv:1606.00099, 2016.

[2] Yao Liang Chung, Maisarah Haji Mohd, A.Swaminathan, and See Keong Lee. Starlikeness of triple integral operators. Far East Journal of Mathematical Sciences, Volume 99, Pages 793-801, 2016.